END-POINT ESTIMATES AND MULTI-PARAMETER PARAPRODUCTS ON HIGHER DIMENSIONAL TORI

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Analogues of multi-parameter multiplier operators on $\mathbb{R}^d$ are defined on the torus $\mathbb{T}^d$. It is shown that these operators satisfy the classical Coifman-Meyer theorem. In addition, $L \log L$ and $L(\log L)^n$ end-point estimates are proved.

BIOGRAPHICAL SKETCH

John Tyler Workman was born September 9, 1981, to parents Jim and Marilyn Workman in Chattanooga, Tennessee. He attended Cleveland High School in Cleveland, Tennessee, graduating as valedictorian in 1999.

He attended the University of Tennessee from 1999 to 2004 and the Royal Melbourne Institute of Technology in Melbourne, Australia, during a semester abroad in 2001. As an undergraduate, John was honored with a Barry M. Goldwater Research Scholarship, along with National Science Foundation and Department of Defense Fellowships for future graduate work. He earned a Bachelor of Science degree in Mathematics in 2004.

In the same year, he began attending Cornell University as a graduate student in pure mathematics. He received the degree of Master of Science in 2007 and Doctor of Philosophy in 2008. To date, he has three publications [17, 18, 19].

John was married in August of 2004 to Shelby Grant-Workman, then Shelby Grant, with whom he has a son, Isaac Workman.

To Shelby and Isaac.

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PREFACE

Consider the classical Marcinkiewicz multiplier operator Λ_m on $\mathbb{R}^d$ defined $\Lambda_m f(x) = \int_{\mathbb{R}^d} m(t)\widehat{f}(t)e^{2\pi itx}\,dt$, where m satisfies a standard Marcinkiewicz-Mihlin-Hörmander type condition [24]. This arises, in part, as a natural extension of the Hilbert transform and Riesz transforms. In 1991, Coifman and Meyer [5] considered a multilinear extension

$$\Lambda_m(f_1,\ldots,f_n)(x) = \int_{\mathbb{R}^{dn}} m(t)\widehat{f}_1(t_1)\cdots\widehat{f}_n(t_n)e^{2\pi ix(t_1+\ldots+t_n)}\,dt,$$

where m, now acting on $\mathbb{R}^{dn}$, satisfies the same kind of condition. This operator is known to map $L^{p_1}\times\ldots\times L^{p_n}\to L^p$ for $1/p_1+\ldots+1/p_n = 1/p$ and $1 < p_j < \infty$. The case when $p \geq 1$ was originally shown by Coifman and Meyer. The general case $p > 1/n$ was settled later in [9, 16].

An important application of this result occurs in non-linear partial differential equations. If $\widehat{D^\alpha f}(t) = |t|^\alpha\widehat{f}(t)$, $\alpha > 0$, is the homogenous derivative, then $\|D^\alpha(fg)\|_r \lesssim \|D^\alpha f\|_p\|g\|_q + \|f\|_p\|D^\alpha g\|_q$ for Schwartz functions f, g, where $1 < p, q < \infty$ and $1/r = 1/p + 1/q$. This inequality was originally proved by Kato and Ponce [14], and can also be established via the Coifman-Meyer theorem (see [26]).

In a more general setting, one can consider an operator $(D_1^\alpha D_2^\beta f)\widehat{\ }(t_1,t_2) = |t_1|^\alpha|t_2|^\beta\widehat{f}(t_1,t_2)$ for $\alpha,\beta > 0$. It is natural to ask, then, is there an analogue to the inequality of Kato and Ponce for this operator. Heuristically, we should have something like $\|D_1^\alpha D_2^\beta(fg)\|_r \lesssim \|D_1^\alpha D_2^\beta f\|_p\|g\|_q + \|D_1^\alpha f\|_p\|D_2^\beta g\|_q + \|D_2^\beta f\|_p\|D_1^\alpha g\|_q + \|f\|_p\|D_1^\alpha D_2^\beta g\|_q$. Attempts to prove this kind of inequality by a Coifman-Meyer type argument lead to a wider class of multipliers m, which behave like the product of two standard multipliers.

Special cases of these multiplier operators had been previously considered by

Christ and Journé [4, 13]. Muscalu et. al. [26] showed in 2004 that this so-called bi-parameter multiplier operator satisfies the same $L^{p_1} \times \ldots \times L^{p_n} \to L^p$ estimates. The original proof for the Coifman-Meyer operator [5, 9, 16] involved the $T1$ theorem, BMO theory, and Carleson measures. Many of these methods, most notably the Calderón-Zygmund decomposition, do not extend to this bi-parameter setting. In [26], an entirely new method based on a strong geometric structure and stopping time arguments is used. This method was further extended in [27] to show that arbitrary multi-parameter multiplier operators satisfy the same bounds.

Another important side-effect of this method is its application to the original Coifman-Meyer operator, giving a much simpler proof. In particular, it establishes the "end-point" estimates of the case when any of the p_j are equal to 1. Here, we have $L^{p_1} \times \ldots \times L^{p_n} \to L^{p,\infty}$. However, in the multi-parameter setting of [26, 27], no such end-point estimates are known.

A natural candidate for such an estimate would involve $L \log L$ spaces, because of how they arise in interpolation results. Naively, it is often believed an operator which maps $L^1 \to L^{1,\infty}$, and also satisfies some L^p result, should take $L \log L$ into L^1. However, it is rarely this straightforward. In [12], Jessen, Marcinkiewicz, and Zygmund showed that if f is in $L \log L$ then Mf (the standard maximal function) is in L^1. But this was only for f, Mf on $[0, 1]$. Wiener [35] improved this by noting that if f, defined on all of $\mathbb{R}^n$, is in $L \log L$, then Mf is locally integrable. Stein [31] showed the converse is true. Indeed, Mf is locally integrable if and only if f is locally in $L \log L$.

Similarly, C. Fefferman [6] examined the role of $L \log L$ as an end-point estimate for the double Hilbert transform and maximal double Hilbert transform. Heuristically, a $L \log L$ to weak-L^1 estimate should be expected. Indeed, this is what is shown, but truncated on the unit square. That is, the maximal double Hilbert transform maps $L\log L([0,1]^2)$ to $L^{1,\infty}([0,1]^2)$.

This problem, that $L\log L$ estimates can only be gained in the compact setting, is rather common. Therefore, the desired end-point estimate for the bi-parameter multiplier operator

$$\Lambda_m : L\log L \times \ldots \times L\log L \to L^{1/n,\infty}$$

is likely to hold only in the compactified sense. However, this leads to an interesting idea: analogues of multiplier operators Λ_m, from the single-parameter case of Coifman and Meyer to the multi-parameter situation of Muscalu et. al., instead defined on the torus $\mathbb{T}^d$. In this setting, that of a probability space, $L\log L$ estimates are often cleaner and conceptually simpler. The establishment and study of the correct operators on tori, and in particular the desire for appropriate end-point estimates, is the focus of this text.

The organization is as follows. Chapter 1 is composed of three parts. The first is a survey of some of the standard analytical tools on the torus. The second part is a series of somewhat technical results concerning special smooth functions. These results are used sporadically throughout the text, but their proofs are similar, so they are presented together. The third part is an interpolation theorem. Chapter 2 covers several different maximal operators, and Chapter 3 deals with a particular square function of Littlewood-Paley type. In Chapter 4, characterizations of $L\log L$ and $L(\log L)^n$ are developed for any probability space, and several important results therein are proved. Chapter 5 introduces and studies single-parameter multipliers, and in particular, analogues of the Marcinkiewicz and Coifman-Meyer multipliers. In Chapter 6, bi-parameter multiplier operators are handled. Chapter 7 is a non-rigorous survey of the proof for multi-parameter multipliers.

Chapter 1
The Circle and Smooth Functions

1.1 Preliminaries

Consider the space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. It has a natural correspondence with a circle of diameter 1 or the interval $[0, 1) \subset \mathbb{R}$, where 0 and 1 are identified. In this way, we can consider Lesbegue measure m on sets $E \subseteq \mathbb{T}$, by considering the corresponding set in $[0, 1)$. Then, $(\mathbb{T}, m)$ is a probability space.

Addition is also naturally defined on $\mathbb{T}$ by the group structure of $\mathbb{R}/\mathbb{Z}$. That is, $x, y \in \mathbb{T}$ can be thought of as elements in $[0, 1)$, and $x + y$ in $\mathbb{T}$ is $(x + y)$ mod-1 in $\mathbb{R}$.

Let $\mathrm{dist}_{\mathbb{R}}(\cdot, \cdot)$ be the Euclidean metric on $\mathbb{R}$, and $\mathrm{dist}_{\mathbb{T}}(\cdot, \cdot)$ the standard metric on $\mathbb{T}$ induced by the geometry of the circle. In particular, if $x, y \in \mathbb{T}$ are thought of as elements in $[0, 1)$, then, $\mathrm{dist}_{\mathbb{T}}(x, y) = \min\{\mathrm{dist}_{\mathbb{R}}(x, y), 1 - \mathrm{dist}_{\mathbb{R}}(x, y)\}$. For sets $A, B \subseteq \mathbb{T}$, let $\mathrm{dist}_{\mathbb{T}}(x, A) = \min\{\mathrm{dist}(x, y) : y \in A\}$ and $\mathrm{dist}_{\mathbb{T}}(A, B) = \min\{\mathrm{dist}_{\mathbb{T}}(x, y) : x \in A, y \in B\}$, as usual.

Functions f acting on $\mathbb{T}$ can simultaneously be thought of as 1-periodic functions acting on $\mathbb{R}$. In this way, we define integration on $(\mathbb{T}, m)$ by

$$\int_{\mathbb{T}} f \, dm = \int_0^1 f(x) \, dx,$$

where the function on the right is defined on $\mathbb{R}$ and integrated over $[0, 1)$. Further, we inherit from $\mathbb{R}$ notions of continuity, differentiability, smoothness, etc.. We will most often consider complex-valued functions, which we write as $f : \mathbb{T} \to \mathbb{C}$. This notation is somewhat misleading, as we allow functions to take infinite values.

For complex scalars α, we will use $|\alpha|$ to denote the modulus or absolute value, and we will denote Lebesgue measure of a set A by $|A|$. This double use should not cause any confusion. We say two sets A, B are disjoint if $|A \cap B| = 0$.

There is a natural notion of intervals in $\mathbb{T}$ as well, that is, connected subsets. We will always use the terminology interval to mean a non-empty, closed interval in $\mathbb{T}$. For simplicity, we allow $\mathbb{T}$ to be considered an interval. We say an interval I is dyadic if $I = [2^{-k}j, 2^{-k}(j+1)]$ for some $j \in \mathbb{Z}$, $k \in \mathbb{N}$. Note that $\mathbb{T}$ itself is not considered a dyadic interval. One can easily show that there is a kind of trichotomy: for any two dyadic intervals either they are equal, one is strictly contained in the other, or they are disjoint.

For any interval I and $0 \leq \alpha \leq 1/|I|$, let αI denote the interval concentric with I which satisfies $|\alpha I| = \alpha |I|$. That is, if $I = \{x : \mathrm{dist}_{\mathbb{T}}(x, x_I) \leq |I|/2\}$, then $\alpha I = \{x : \mathrm{dist}_{\mathbb{T}}(x, x_I) \leq \alpha |I|/2\}$. For integers n, let $I^n = I + n|I|$, the interval gained by shifting n steps of length $|I|$.

Finally, we will use the somewhat standard notation $A \lesssim B$ to mean that there is some "universal" constant C such that $A \leq C \cdot B$. We will write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. It will be our attempt throughout to make as clear as possible precisely what these unspoken constants depend on.

1.2 Analysis on $\mathbb{T}$

Many of the fundamental analytical tools which we use on $\mathbb{R}^n$ can be easily extended to $\mathbb{T}$. Katznelson [15] gives a comprehensive introduction to this topic.

Considering the probability space $(\mathbb{T}, m)$, we can define $\|f\|_p = (\int_{\mathbb{T}} |f|^p \, dm)^{1/p}$ for $0 < p < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup}_{\mathbb{T}} |f|$ as normal. Then, the spaces $L^p(\mathbb{T})$ of functions for which $\|f\|_p < \infty$ are Banach spaces for $1 \leq p \leq \infty$. Similarly, we define weak-$L^p(\mathbb{T})$ or $L^{p,\infty}(\mathbb{T})$ as the functions for which

$$\|f\|_{p,\infty} := \sup_{\lambda > 0} \lambda \left| \left\{ x \in \mathbb{T} : |f(x)| > \lambda \right\} \right|^{1/p} < \infty.$$

Note, $\|\cdot\|_{p,\infty}$ is only a quasi-norm, in that it does not always satisfy the triangle inequality. However, it is true that $|f| \leq |g|$ a.e. implies $\|f\|_{p,\infty} \leq \|g\|_{p,\infty}$ and $f_n \uparrow |f|$ a.e. implies $\|f_n\|_{p,\infty} \uparrow \|f\|_{p,\infty}$.

Denote the L^2 inner product by $\langle\cdot,\cdot\rangle$, i.e., $\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$, where $\overline{g}$ is the complex conjugate. It will be our practice, when studying an operator T, to write $T : L^p \to L^p$ or that T maps L^p to L^p, when it is actually meant maps boundedly. In particular, that there is some constant C so that $\|Tf\|_p \leq C\|f\|_p$ for all f.

For $f \in L^1(\mathbb{T})$ we define the Fourier coefficients of f by

$$\widehat{f}(n) = \int_{\mathbb{T}} f(x)e^{-2\pi inx}\,dx$$

for all $n \in \mathbb{Z}$. It is easily shown [15] that the usual properties hold. In particular, this operation is linear, and if we define convolution as

$$(f * g)(x) = \int_{\mathbb{T}} f(y)g(x-y)\,dy,$$

then $\widehat{(f * g)}(n) = \widehat{f}(n)\widehat{g}(n)$ for all n. Further, we have a version of Plancherel's theorem: for $f, g \in L^2(\mathbb{T})$

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,\overline{\widehat{g}(n)}$$

or equivalently,

$$\int_{\mathbb{T}} f(x)g(x)\,dx = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\widehat{g}(-n).$$

It is also well-known that if a function f is smooth

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inx}.$$

Recall that a function $f : \mathbb{R} \to \mathbb{C}$ is a Schwartz function [33, 34] if it is infinitely differentiable and $\sup_{\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty$ for all integers $k, l \geq 0$. For a Schwartz function f, define its periodization by

$$F(x) = \sum_{j \in \mathbb{Z}} f(x + j).$$

This function is clearly 1-periodic, so we may think of F as a function on $\mathbb{T}$. This sum converges absolutely for all x, which follows because $|f(x+j)| \leq C|x+j|^{-2}$ for some C is guaranteed by the Schwartz condition, . For $h \neq 0$, we have by the mean value theorem that $\frac{1}{h}[F(x+h) - F(x)] = \sum_j f'(x_{j,h})$, where $x_{j,h}$ is some number between $x + j$ and $x + j + h$. Using the Schwartz property again, we can apply the dominated convergence theorem to let $h \to 0$ and see F is differentiable, with $F'(x) = \sum_j f'(x + j)$. Iterating this, we find that F is smooth (infinitely differentiable) and $F^{(l)}$ is simply the periodization of $f^{(l)}$.

Furthermore,

$$\begin{aligned} \widehat{F}(n) &= \int_{\mathbb{T}} F(x) e^{-2\pi i n x}\, dx = \int_0^1 F(x) e^{-2\pi i n x}\, dx \\ &= \sum_j \int_0^1 f(x+j) e^{-2\pi i n x}\, dx = \int_{\mathbb{R}} f(x) e^{-2\pi i n x}\, dx = \widehat{f}(n). \end{aligned}$$

That is, the Fourier coefficients of F coincide with the Fourier transform of f on the integers.

Finally, we can define $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, the d-fold product of $\mathbb{T}$. Lebesgue measure can be attained from $\mathbb{R}^d$ just as before, or as the appropriate product measure, so that $(\mathbb{T}^d, m)$ is a probability space. Functions $f : \mathbb{T}^d \to \mathbb{C}$ can be thought of as functions on $\mathbb{R}^d$ which are 1-periodic in each coordinate, and integration is defined as before. The Fourier coefficients $\widehat{f}(n_1, \ldots, n_d) = \int_{\mathbb{T}^d} f(\vec{x}) e^{-2\pi i \vec{n} \cdot \vec{x}}\, d\vec{x}$ are defined in a natural way, and all the normal results hold.

1.3 Bump Functions

Our first goal will be to generate a sequence of smooth functions whose Fourier coefficients are a kind of "partition of unity." It turns out the easiest way to do this is to first create the functions on $\mathbb{R}$ and then periodize.

Theorem 1.1. *There are Schwartz functions $\theta_k^1, \theta_k^2 : \mathbb{R} \to \mathbb{C}$, $k \in \mathbb{Z}$, and constants $C_m > 0$, $m \in \mathbb{N}$, so that*

$$\sum_{k \in \mathbb{Z}} \widehat{\theta_k^1}(t)\widehat{\theta_k^2}(-t) = \chi_{\mathbb{R}-0}(t),$$

$$\operatorname{supp}(\widehat{\theta_k^1}) \subseteq [-2^{k-2}, -2^{k-4}] \cup [2^{k-4}, 2^{k-2}], \ \widehat{\theta_k^2}(0) = 0,$$

$$|\theta_k^1(x)|, |\theta_k^2(x)| \leq 2^k C_m \Big(1 + 2^k \operatorname{dist}_{\mathbb{R}}(x, [0, 2^{-k}])\Big)^{-m} \quad \textit{for all } x \in \mathbb{R},\, m \in \mathbb{N},$$

$$|\theta_k^{1'}(x)|, |\theta_k^{2'}(x)| \leq 4^k C_m \Big(1 + 2^k \operatorname{dist}_{\mathbb{R}}(x, [0, 2^{-k}])\Big)^{-m} \quad \textit{for all } x \in \mathbb{R},\, m \in \mathbb{N}.$$

Proof. Choose a Schwartz function $\alpha : \mathbb{R} \to \mathbb{C}$ so that $\widehat{\alpha} = 1$ on $[-1/8, 1/8]$ and $\operatorname{supp}(\widehat{\alpha}) \subseteq [-1/4, 1/4]$. Define $\widehat{\theta^1}(t) = \widehat{\alpha}(t) - \widehat{\alpha}(2t)$. Let $\widehat{\theta_k^1}(t) = \widehat{\theta^1}(2^{-k}t)$ for all $k \in \mathbb{Z}$.

Fix $t \neq 0$. Choose any $N \in \mathbb{N}$ so that $|t| \leq 2^{N-3}$ and $|t| > 2^{-N-3}$. Then,

$$\sum_{k=-N}^{N} \widehat{\theta_k^1}(t) = \Big(\widehat{\alpha}(2^N t) - \widehat{\alpha}(2^{N+1} t)\Big) + \Big(\widehat{\alpha}(2^{N-1} t) - \widehat{\alpha}(2^N t)\Big) + \ldots +$$

$$\Big(\widehat{\alpha}(2^{-N} t) - \widehat{\alpha}(2^{-N+1} t)\Big)$$

$$= \widehat{\alpha}(2^{-N} t) - \widehat{\alpha}(2^{N+1} t) = 1 - 0 = 1.$$

As this holds for all N big enough, and t is arbitrary, it follows that $\sum_k \widehat{\theta_k^1}(t) = 1$ for all $t \neq 0$. On the other hand, as $\widehat{\theta_k^1}(0) = \widehat{\theta^1}(0) = 0$ for all k, it is clear the sum is 0 at $t = 0$.

Fix $k \in \mathbb{Z}$. Let $|t| \leq 2^{k-4}$. Then $|2^{-k}t|, |2^{-k+1}t| \leq 1/8$. This implies $\widehat{\theta}_k^1(t) = \widehat{\theta}^1(2^{-k}t) = \widehat{\alpha}(2^{-k}t) - \widehat{\alpha}(2^{-k+1}t) = 1 - 1 = 0$. Similarly, if $|t| > 2^{k-2}$, then $|2^{-k+1}t|, |2^{-k}t| > 1/4$, and $\widehat{\theta}_k^1(t) = \widehat{\alpha}(2^{-k}t) - \widehat{\alpha}(2^{-k+1}t) = 0 - 0 = 0$. That is, $\mathrm{supp}(\widehat{\theta}_k^1) \subseteq [-2^{k-2}, -2^{k-4}] \cup [2^{k-4}, 2^{k-2}]$.

Choose a Schwartz function θ^2 so that $\widehat{\theta}^2 = 1$ on $[-1/8, -1/16] \cup [1/16, 1/8]$ and is supported away from 0. Define $\widehat{\theta}_k^2(t) = \widehat{\theta}^2(2^{-k}t)$. Then, $\widehat{\theta}_k^2(0) = 0$ and $\widehat{\theta}_k^2 = 1$ on $[-2^{k-2}, -2^{k-4}] \cup [2^{k-4}, 2^{k-2}] \supseteq \mathrm{supp}(\widehat{\theta}_k^1)$, so that

$$\sum_{k \in \mathbb{Z}} \widehat{\theta}_k^1(t) \widehat{\theta}_k^2(-t) = \sum_{k \in \mathbb{Z}} \widehat{\theta}_k^1(t) = \chi_{\mathbb{R}-0}(t).$$

Finally, note that $\theta_k^i(x) = 2^k \theta^i(2^k x)$ for $i = 1, 2$. As θ^i and $\theta^{i'}$ are Schwartz functions and $(1 + \mathrm{dist}_{\mathbb{R}}(x, [0,1]))^m$ has polynomial growth, we can choose C_m so that $|\theta^i(x)|, |\theta^{i'}(x)| \leq C_m(1 + \mathrm{dist}_{\mathbb{R}}(x, [0,1]))^{-m}$ for all x and m and $i = 1, 2$. Then, $|\theta_k^i(x)| = 2^k |\theta^i(2^k x)| \leq 2^k C_m (1 + \mathrm{dist}_{\mathbb{R}}(2^k x, [0,1]))^{-m} = 2^k C_m (1 + 2^k \,\mathrm{dist}_{\mathbb{R}}(x, [0, 2^{-k}]))^{-m}$. By the same argument, $|\theta_k^{i'}(x)| = 4^k |\theta'(2^k)| \leq 4^k C_m (1 + 2^k \,\mathrm{dist}_{\mathbb{R}}(x, [0, 2^{-k}]))^{-m}$. □

Claim 1.2. *Fix $j, k \in \mathbb{N}$ and define $f(t) = j(1 + 2^k \min(t, 1-t)) - 2^k(t + j - 1)$. For any $t \in [0,1]$, $f(t) \leq 1$.*

Proof. For $t \in [0, 1/2]$, $f(t) = j(1 + 2^k t) - 2^k(t + j - 1)$, which is an increasing linear function in t. Indeed, $f'(t) = j2^k - 2^k \geq 0$. For $t \in [1/2, 1]$, $f(t) = j(1 + 2^k(1-t)) - 2^k(t + j - 1)$, which is a decreasing linear function in t, as $f'(t) = -j2^k - 2^k < 0$. Thus, $\max_{x \in [0,1]} f(t) = f(1/2) = j(1 + 2^{k-1}) - 2^k(j - 1/2) = j + 2^{k-1} - j2^{k-1} \leq 1$. This last inequality follows as $a + b - ab \leq 1$ for any positive integers a, b, which is easily shown through induction. □

Lemma 1.3. *Let $\theta_k : \mathbb{R} \to \mathbb{C}$ be a Schwartz function and $\psi_k : \mathbb{T} \to \mathbb{C}$ its periodization. If*

$$|\theta_k(x)| \le C_m 2^k \big(1 + 2^k \operatorname{dist}_{\mathbb{R}}(x, [0, 2^{-k}])\big)^{-m} \qquad \textit{and}$$
$$|\theta_k'(x)| \le C_m 4^k \big(1 + 2^k \operatorname{dist}_{\mathbb{R}}(x, [0, 2^{-k}])\big)^{-m},$$

then there exist constants C_m' so that

$$|\psi_k(x)| \le C_m' 2^k \big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])\big)^{-m} \qquad \textit{and}$$
$$|\psi_k'(x)| \le C_m' 4^k \big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])\big)^{-m}.$$

Proof. Fix $x \in [0, 1)$. Clearly, $\operatorname{dist}_{\mathbb{R}}(x, [0, 2^{-k}]) \ge \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])$. Hence, $|\theta_k(x)| \le C_m 2^k (1 + \operatorname{dist}_{\mathbb{R}}(x, [0, 2^{-k}]))^{-m} \le C_m 2^k (1 + \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}]))^{-m}$. For any $j \in \mathbb{N}$, note $\operatorname{dist}_{\mathbb{R}}(x+j, [0, 2^{-k}]) \ge \operatorname{dist}_{\mathbb{R}}(x, [0, 2^{-k}]) + j - 1$. Set $t = \operatorname{dist}_{\mathbb{R}}(x, [0, 2^{-k}])$, and observe that $t \in [0, 1]$ and $\operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}]) \le \min(t, 1-t)$. Thus, by Claim 1.2,

$$\begin{aligned} j\big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])\big) &\le j\big(1 + 2^k \min(t, 1-t)\big) = f(t) + 2^k(t + j - 1) \\ &\le 1 + 2^k(t + j - 1) \le 1 + 2^k \operatorname{dist}_{\mathbb{R}}(x + j, [0, 2^{-k}]). \end{aligned}$$

Therefore, we see that for any integer $m > 1$,

$$\begin{aligned} \sum_{j=1}^{\infty} |\theta_k(x+j)| &\le \sum_{j=1}^{\infty} C_m 2^k \Big(1 + \operatorname{dist}_{\mathbb{R}}(x + j, [0, 2^{-k}])\Big)^{-m} \\ &\le C_m 2^k \sum_{j=1}^{\infty} j^{-m} \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])\Big)^{-m} \\ &\le 2 C_m 2^k \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])\Big)^{-m} \end{aligned}$$

Similarly, $\operatorname{dist}_{\mathbb{R}}(x-1, [0, 2^{-k}]) = \operatorname{dist}_{\mathbb{R}}(x-1, 0) = \operatorname{dist}_{\mathbb{R}}(x, 1) \ge \operatorname{dist}_{\mathbb{T}}(x, 1) \ge \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])$. Therefore, $|\theta_k(x-1)| \le C_m 2^k (1 + \operatorname{dist}_{\mathbb{R}}(x-1, [0, 2^{-k}]))^{-m} \le C_m 2^k (1 + 2^k \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}]))^{-m}$, and for $j \in \mathbb{N}$, we have $\operatorname{dist}_{\mathbb{R}}(x-j, [0, 2^{-k}]) = \operatorname{dist}_{\mathbb{R}}(x-1, [0, 2^{-k}]) + j - 1$. Set $t = \operatorname{dist}_{\mathbb{R}}(x-1, [0, 2^{-k}])$, and again observe that

$t \in [0,1]$ and $\operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}]) \leq \min(t, 1-t)$. Using the claim as before, it follows that $j(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}])) \leq 1 + 2^k \operatorname{dist}_{\mathbb{R}}(x-j,[0,2^{-k}])$. Thus, for $m > 1$,

$$\begin{aligned}\sum_{j=2}^{\infty} |\theta_k(x-j)| &\leq \sum_{j=2}^{\infty} C_m 2^k \Big(1 + \operatorname{dist}_{\mathbb{R}}(x-j,[0,2^{-k}])\Big)^{-m} \\ &\leq C_m 2^k \sum_{j=2}^{\infty} j^{-m} \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}])\Big)^{-m} \\ &\leq 2C_m 2^k \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}])\Big)^{-m}\end{aligned}$$

Hence,

$$\begin{aligned}|\psi_k(x)| &\leq \sum_{j\in\mathbb{Z}} |\theta_k(x+j)| \\ &= |\theta_k(x)| + |\theta_k(x-1)| + \sum_{j=1}^{\infty} |\theta_k(x+j)| + \sum_{j=2}^{\infty} |\theta_k(x-j)| \\ &\leq (C_m + C_m + 2C_m + 2C_m) 2^k \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}])\Big)^{-m}.\end{aligned}$$

Now, this holds for all $m > 1$. But, of course, the $m = 1$ case follows as $|\psi_k(x)| \leq 6C_2 2^k (1+2^k \operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}]))^{-2} \leq 6C_2 2^k (1+2^k \operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}]))^{-1}$. The condition on ψ_k' is proven in exactly the same manner. Thus, the statement holds with $C_m' = 6C_m$ for $m > 1$ and $C_1' = 6C_2$. □

Theorem 1.4. *There are smooth functions $\psi_k^1, \psi_k^2 : \mathbb{T} \to \mathbb{C}$, $k \in \mathbb{N}$, and constants $C_m > 0$, $m \in \mathbb{N}$, so that*

$$\sum_{k=1}^{\infty} \widehat{\psi_k^1}(n) \widehat{\psi_k^2}(-n) = \chi_{\mathbb{Z}-0}(n),$$

$$\operatorname{supp}(\widehat{\psi_k^1}) \subseteq [-2^{k-2}, -2^{k-4}] \cup [2^{k-4}, 2^{k-2}], \ \widehat{\psi_k^2}(0) = 0,$$

$$|\psi_k^1(x)|, |\psi_k^2(x)| \leq 2^k C_m \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}])\Big)^{-m} \quad \textit{for all } x \in \mathbb{T},\ m \in \mathbb{N},$$

$$|\psi_k^{1'}(x)|, |\psi_k^{2'}(x)| \leq 4^k C_m \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x,[0,2^{-k}])\Big)^{-m} \quad \textit{for all } x \in \mathbb{T},\ m \in \mathbb{N}.$$

Proof. Let $\theta_k^1, \theta_k^2 : \mathbb{R} \to \mathbb{C}$, $k \in \mathbb{Z}$, be the functions guaranteed by Theorem 1.1, and ψ_k^1, ψ_k^2 their respective periodizations. As $\widehat{\theta_k^i}(n) = \widehat{\psi_k^i}(n)$, it follows $\widehat{\psi_k^2}(0) = 0$ and $\mathrm{supp}(\widehat{\psi_k^1}) \subseteq [-2^{k-2}, -2^{k-4}] \cup [2^{k-4}, 2^{k-2}]$. From this, we see for any integer n that $\widehat{\psi_k^1}(n)\widehat{\psi_k^2}(-n) = 0$ for $k \leq 0$. Thus,

$$\sum_{k=1}^{\infty} \widehat{\psi_k^1}(n)\widehat{\psi_k^2}(-n) = \sum_{k\in\mathbb{Z}} \widehat{\theta}_k^1(n)\widehat{\theta}_k^2(-n) = \chi_{\mathbb{Z}-0}(n).$$

Finally, the inequalities on ψ_k^i and $\psi_k^{i'}$ follow from Theorem 1.1 and Lemma 1.3. $\square$

Theorem 1.5. *There are Schwartz functions $\theta_k^{a,i} : \mathbb{R} \to \mathbb{C}$, $1 \leq a, i \leq 3$, $k \in \mathbb{Z}$, and constants $C_m > 0$, $m \in \mathbb{N}$, so that*

$$\sum_{a=1}^{3}\sum_{k\in\mathbb{Z}} \widehat{\theta_k^{a,1}}(t_1)\widehat{\theta_k^{a,2}}(t_2)\widehat{\theta_k^{a,3}}(-t_1-t_2) = \chi_{\mathbb{R}^2-(0,0)}(t_1,t_2)$$

$$\mathrm{supp}(\widehat{\theta_k^{a,i}}) \subseteq [-2^{k-2}, -2^{k-10}] \cup [2^{k-10}, 2^{k-2}] \quad \text{for } a \neq i,$$

$$\mathrm{supp}(\widehat{\theta_k^{a,i}}) \subseteq [-2^{k-2}, 2^{k-2}] \quad \text{for } a = i,$$

$$|\theta_k^{a,i}(x)| \leq 2^k C_m\Big(1 + 2^k \,\mathrm{dist}_{\mathbb{R}}(x, [0, 2^{-k}])\Big)^{-m} \quad \text{for all } x \in \mathbb{R},\ m \in \mathbb{N},$$

$$|\theta_k^{a,i'}(x)| \leq 4^k C_m\Big(1 + 2^k \,\mathrm{dist}_{\mathbb{R}}(x, [0, 2^{-k}])\Big)^{-m} \quad \text{for all } x \in \mathbb{R},\ m \in \mathbb{N}.$$

Proof. Similar to the proof of Theorem 1.1, start with a Schwartz bump $\widehat{\alpha}$ which is identically 1 on $[-1/64, 1/64]$ and supported in $[-1/32, 1/32]$. Set $\widehat{\beta}(t) = \widehat{\alpha}(t) - \widehat{\alpha}(2t)$. Define $\widehat{\beta_k^1}(t) = \widehat{\beta}(2^{-k}t)$ and $\widehat{\beta_k^2}(t) = \widehat{\alpha}(2^{-k+3}t)$. Set $\widehat{\beta_k^3}(t) = \sum_{j=k-2}^{k+2} \widehat{\beta_j^1}(t)$.

By construction of α, we can see $\mathrm{supp}(\widehat{\beta_k^2}) \subseteq [-2^{k-8}, 2^{k-8}]$. By an argument similar to that in Theorem 1.1, $\mathrm{supp}(\widehat{\beta_k^1}) \subseteq [-2^{k-5}, -2^{k-7}] \cup [2^{k-7}, 2^{k-5}]$. Thus, $\mathrm{supp}(\widehat{\beta_k^3}) \subseteq [-2^{k-3}, -2^{k-9}] \cup [2^{k-9}, 2^{k-3}]$.

Fix $t \in \mathbb{R}$, $t \neq 0$, and choose $N \in \mathbb{N}$ so that $|t| > 2^{-N-6}$. Then, $|2^{N+1}t| > 1/32$ and by the same telescoping argument as before

$$\sum_{j=-N}^{k-3} \widehat{\beta_j^1}(t) = \left(\widehat{\alpha}(2^N t) - \widehat{\alpha}(2^{N+1}t)\right) + \ldots + \left(\widehat{\alpha}(2^{-k+3}t) - \widehat{\alpha}(2^{-k+3}t)\right)$$

$$= \widehat{\alpha}(2^{-k+3}t) - \widehat{\alpha}(2^{N+1}t) = \widehat{\alpha}(2^{-k+3}t) = \widehat{\beta_k^2}(t).$$

As N and t are arbitrary, we have that $\sum_{j<k-2} \widehat{\beta_j^1}(t) = \widehat{\beta_k^2}(t)$ for $t \neq 0$. By the same argument used in Theorem 1.1, $\sum \widehat{\beta_k^1}(t) = 1$ for all $t \neq 0$.

Fix $t_1, t_2 \in \mathbb{R}$, both non-zero. Then,

$$\begin{aligned}
1 &= \left(\sum_{k_1 \in \mathbb{Z}} \widehat{\beta_{k_1}^1}(t_1)\right)\left(\sum_{k_2 \in \mathbb{Z}} \widehat{\beta_{k_2}^1}(t_2)\right) \\
&= \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 > k_1+2} \widehat{\beta_{k_1}^1}(t_1)\widehat{\beta_{k_2}^1}(t_2) + \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 < k_1-2} \widehat{\beta_{k_1}^1}(t_1)\widehat{\beta_{k_2}^1}(t_2) \\
&\quad + \sum_{k_1 \in \mathbb{Z}} \sum_{k_2=k_1-2}^{k_1+2} \widehat{\beta_{k_1}^1}(t_1)\widehat{\beta_{k_2}^1}(t_2) \\
&= \sum_{k \in \mathbb{Z}} \widehat{\beta_k^2}(t_1)\widehat{\beta_k^1}(t_2) + \sum_{k \in \mathbb{Z}} \widehat{\beta_k^1}(t_1)\widehat{\beta_k^2}(t_2) + \sum_{k \in \mathbb{Z}} \widehat{\beta_k^1}(t_1)\widehat{\beta_k^3}(t_2).
\end{aligned}$$

On the other hand, $\widehat{\beta_k^2}(0) = \widehat{\alpha}(0) = 1$. Hence, in the $t_1 = 0$ case, we see that for any $t_2 \neq 0$

$$\sum_{k \in \mathbb{Z}} \widehat{\beta_k^2}(0)\widehat{\beta_k^1}(t_2) + \sum_{k \in \mathbb{Z}} \widehat{\beta_k^1}(0)\widehat{\beta_k^2}(t_2) + \sum_{k \in \mathbb{Z}} \widehat{\beta_k^1}(0)\widehat{\beta_k^3}(t_2) = \sum_{k \in \mathbb{Z}} \widehat{\beta_k^1}(t_2) = 1.$$

The $t_2 = 0$ case is symmetrical. We note that when $t_1 = t_2 = 0$, the triple sum is equal to 0. Hence,

$$\sum_{k \in \mathbb{Z}} \widehat{\beta_k^2}(t_1)\widehat{\beta_k^1}(t_2) + \sum_{k \in \mathbb{Z}} \widehat{\beta_k^1}(t_1)\widehat{\beta_k^2}(t_2) + \sum_{k \in \mathbb{Z}} \widehat{\beta_k^1}(t_1)\widehat{\beta_k^3}(t_2) = \chi_{\mathbb{R}^2-(0,0)}(t_1, t_2).$$

Define $\beta_k^1 = \theta_k^{1,2} = \theta_k^{2,1} = \theta_k^{3,1}$, $\beta_k^2 = \theta_k^{1,1} = \theta_k^{2,2}$, and $\beta_k^3 = \theta_k^{3,2}$ and observe

$$\sum_{k\in\mathbb{Z}} \widehat{\theta_k^{1,1}}(t_1)\widehat{\theta_k^{1,2}}(t_2) + \sum_{k\in\mathbb{Z}} \widehat{\theta_k^{2,1}}(t_1)\widehat{\theta_k^{2,2}}(t_2) + \sum_{k\in\mathbb{Z}} \widehat{\theta_k^{3,1}}(t_1)\widehat{\theta_k^{3,2}}(t_2) = \chi_{\mathbb{R}^2-(0,0)}(t_1,t_2).$$

Choose a Schwartz function γ^1 supported in $[-2^{-3},-2^{-9}]\cup[2^{-9},2^{-3}]$ and identically 1 on $[-2^{-4},-2^{-8}]\cup[2^{-8},2^{-4}]$. Let $\widehat{\gamma_k^1}(t) = \widehat{\gamma^1}(2^{-k}t)$. Then, $\mathrm{supp}(\widehat{\gamma_k^1}) \subseteq [-2^{k-3},-2^{k-9}]\cup[2^{k-9},2^{k-3}]$ and $\widehat{\gamma_k^1} = 1$ on $[-2^{k-4},-2^{k-8}]\cup[2^{k-8},2^{k-4}]$. Now, if $2^{k-7} \le |t_1| \le 2^{k-5}$ and $|t_2| \le 2^{k-8}$, then $2^{k-8} \le |t_1+t_2| \le 2^{k-4}$. Hence, $\widehat{\gamma_k^1}(-t_1-t_2) = 1$ for such t_1, t_2. In particular,

$$\begin{aligned}\widehat{\theta_k^{2,1}}(t_1)\widehat{\theta_k^{2,2}}(t_2)\widehat{\gamma_k^1}(-t_1-t_2) &= \widehat{\beta_k^1}(t_1)\widehat{\beta_k^2}(t_2)\widehat{\gamma_k^1}(-t_1-t_2)\\ &= \widehat{\beta_k^1}(t_1)\widehat{\beta_k^2}(t_2) = \widehat{\theta_k^{2,1}}(t_1)\widehat{\theta_k^{2,2}}(t_2).\end{aligned}$$

By symmetry,

$$\widehat{\theta_k^{1,1}}(t_1)\widehat{\theta_k^{1,2}}(t_2)\widehat{\gamma_k^1}(-t_1-t_2) = \widehat{\theta_k^{1,1}}(t_1)\widehat{\theta_k^{1,2}}(t_2).$$

Set $\theta_k^{1,3} = \theta_k^{2,3} = \gamma^1$.

Similarly, if we choose a Schwartz function γ^2 so that $\widehat{\gamma^2}$ is supported in $[-1/4,1/4]$ and identically 1 on $[-1/8-1/32, 1/8+1/32]$, and let $\widehat{\gamma_k^2}(t) = \widehat{\gamma^2}(2^{-k}t)$, then $\widehat{\gamma_k^2}$ is supported in $[-2^{k-2},2^{k-2}]$ and identically 1 on $[-2^{k-3}-2^{k-5}, 2^{k-5}+2^{k-3}]$. Thus,

$$\widehat{\theta_k^{3,1}}(t_1)\widehat{\theta_k^{3,2}}(t_2)\widehat{\gamma_k^2}(-t_1-t_2) = \widehat{\theta_k^{3,1}}(t_1)\widehat{\theta_k^{3,2}}(t_2).$$

Set $\theta_k^{3,3} = \gamma_k^2$. It is now clear that the appropriate sum condition holds.

As $\beta_0^1, \beta_0^2, \beta_0^3, \gamma_0^1, \gamma_0^2$ are all Schwartz bumps, we can choose constants C_m so that $|\beta_0^i|, |\beta_0^{i'}|, |\gamma_0^j|, |\gamma_0^{j'}| \le C_m(1+\mathrm{dist}_{\mathbb{R}}(x,[0,1]))^{-m}$ for $i = 1,2,3$ and $j = 1,2$. Then, as in the proof of Theorem 1.1, $|\theta_k^{a,i}(x)| \le C_m 2^k(1+2^k\,\mathrm{dist}_{\mathbb{R}}(x,[0,2^{-k}]))^{-m}$ and $|\theta_k^{a,i'}(x)| \le 4^k C_m(1+2^k\,\mathrm{dist}_{\mathbb{R}}(x,[0,2^{-k}]))^{-m}$. □

Theorem 1.6. *There are smooth functions* $\psi_k^{a,i} : \mathbb{T} \to \mathbb{C}$, $1 \le a, i \le 3$, $k \in \mathbb{N}$, *and constants* $C_m > 0$, $m \in \mathbb{N}$, *so that*

$$\sum_{a=1}^{3} \sum_{k=1}^{\infty} \widehat{\psi_k^{a,1}}(n_1)\widehat{\psi_k^{a,2}}(n_2)\widehat{\psi_k^{a,3}}(-n_1 - n_2) = \chi_{\mathbb{Z}^2-(0,0)}(n_1, n_2)$$

$$\operatorname{supp}(\widehat{\psi_k^{a,i}}) \subseteq [-2^{k-2}, -2^{k-10}] \cup [2^{k-10}, 2^{k-2}] \quad \textit{for } a \neq i,$$

$$\operatorname{supp}(\widehat{\psi_k^{a,i}}) \subseteq [-2^{k-2}, 2^{k-2}] \quad \textit{for } a = i,$$

$$|\psi_k^{a,i}(x)| \le 2^k C_m \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])\Big)^{-m} \quad \textit{for all } x \in \mathbb{T},\ m \in \mathbb{N},$$

$$|\psi_k^{a,i\prime}(x)| \le 4^k C_m \Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x, [0, 2^{-k}])\Big)^{-m} \quad \textit{for all } x \in \mathbb{T},\ m \in \mathbb{N}.$$

Proof. Let $\theta_k^{a,i}$ be the functions guaranteed by Theorem 1.5, and let $\psi_k^{a,i}$ be their respective periodizations. Noting that $\widehat{\psi_k^{a,i}}(n) = 0$ for all integers $n \neq 0$ when $k \le 0$, everything follows immediately from Theorem 1.5. □

1.4 Adapted Families

Definition. We say a smooth function $\varphi : \mathbb{T} \to \mathbb{C}$ is adapted to an interval I with constants $C_m > 0$, $m \in \mathbb{N}$, if

$$|\varphi(x)| \le C_m \left(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x, I)}{|I|}\right)^{-m} \text{ for all } x \in \mathbb{T}, m \in \mathbb{N},$$

$$|\varphi'(x)| \le C_m \frac{1}{|I|} \left(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x, I)}{|I|}\right)^{-m} \text{ for all } x \in \mathbb{T}, m \in \mathbb{N}.$$

A family of smooth functions $\varphi_I : \mathbb{T} \to \mathbb{C}$, indexed by the dyadic intervals, is called an adapted family if each φ_I is adapted to I with the same universal constants. We say $\{\varphi_I\}_I$ is a 0-mean adapted family if it is an adapted family, with the additional property that $\int_{\mathbb{T}} \varphi_I \, dm = 0$ for all I.

The first question we should address is whether such a family exists. Take either ψ_k^1 or ψ_k^2 from Theorem 1.4. We will write ψ_k for simplicity. For each dyadic interval $I = [2^{-k}j, 2^{-k}(j+1)]$, define $\varphi_I(x) = 2^{-k}\psi_k(x - 2^{-k}j)$. Then,

$$\begin{aligned}
|\varphi_I(x)| &= |2^{-k}\psi_k(x - 2^{-k}j)| \\
&\leq C_m\Big(1 + 2^k \operatorname{dist}_{\mathbb{T}}(x - 2^{-k}j, [0, 2^{-k}])\Big)^{-m} \\
&= C_m\left(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x, I)}{|I|}\right)^{-m}.
\end{aligned}$$

Similarly, $|\varphi_I'(x)| = |2^{-k}\psi_k'(x - 2^{-k}j)| \leq C_m\frac{1}{|I|}(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|})^{-m}$. Therefore, we have established a way to generate adapted families. In fact, this is a 0-mean adapted family, as $\widehat{\psi_k^1}(0) = \widehat{\psi_k^2}(0) = 0$. However, there are adapted families with even more specific properties.

Theorem 1.7. *There exists a 0-mean adapted family $\{\varphi_I\}_I$ and a constant $a > 0$ so that $|\varphi_I| \geq a\chi_I$ for all I.*

Proof. Choose a Schwartz function $\alpha : \mathbb{R} \to \mathbb{C}$ so that $\widehat{\alpha} = 1$ on $[-1/2, 1/2]$, $\operatorname{supp}(\widehat{\alpha}) \subseteq [-1, 1]$, and $s = |\alpha(0)| > 0$. By continuity, choose an integer $k_0 \geq 0$ so that $|x| \leq 2^{-k_0}$ implies $|\alpha(x) - \alpha(0)| < s/4$. Then, for $x \in [0, 2^{-k_0}]$, we have $|\alpha(x)| - s \leq |\alpha(x) - \alpha(0)| < s/4$ or $|\alpha(x)| < \frac{5s}{4}$. Similarly, $s - |\alpha(x)| \leq |\alpha(x) - \alpha(0)| < s/4$ or $|\alpha(x)| > \frac{3s}{4}$. Set $\beta(x) = \alpha(2^{-k_0}x)$, giving $\frac{3s}{4} < |\beta(x)| < \frac{5s}{4}$ for all $x \in [0, 1]$. Define $\widehat{\theta}(x) = \widehat{\beta}(x) - \widehat{\beta}(2x)$ and $\widehat{\theta}_k(x) = \widehat{\theta}(2^{-k}x)$ for all $k \in \mathbb{N}$.

Now, $\theta_k(x) = 2^k\theta(2^kx)$ and $\theta(x) = \beta(x) - \frac{1}{2}\beta(\frac{1}{2}x)$. For any $x \in [0, 1]$, we see $|\theta(x)| \geq |\beta(x)| - \frac{1}{2}|\beta(\frac{1}{2}x)| \geq \frac{3s}{4} - \frac{5s}{8} = \frac{s}{8} =: c$. Thus, for any $x \in [0, 2^{-k}]$, we have $|\theta_k(x)| = 2^k|\theta(2^kx)| \geq c2^k$. Namely, $|\theta_k| \geq c2^k\chi_{[0,2^{-k}]}$. It is easily seen that $\widehat{\theta}_k(0) = \widehat{\theta}(0) = \widehat{\beta}(0) - \widehat{\beta}(0) = 0$.

Note θ and θ' are Schwartz functions and $(1 + \operatorname{dist}_{\mathbb{R}}(x, [0, 1]))^m$ has polynomial growth. Choose C_m so that $|\theta(x)|, |\theta'(x)| \leq C_m(1 + \operatorname{dist}_{\mathbb{R}}(x, [0, 1]))^{-m}$ for all x

and m. By the same manipulations as before, this implies $|\theta_k(x)| = 2^k|\theta(2^k x)| \leq 2^k C_m(1 + \mathrm{dist}_{\mathbb{R}}(2^k x, [0,1]))^{-m} = 2^k C_m(1 + 2^k \mathrm{dist}_{\mathbb{R}}(x, [0, 2^{-k}]))^{-m}$, and $|\theta_k'(x)| = 4^k|\theta'(2^k)| \leq 4^k C_m(1 + 2^k \mathrm{dist}_{\mathbb{R}}(x, [0, 2^{-k}]))^{-m}$.

Let ψ_k be the periodization of θ_k. As $\widehat{\psi}_k(0) = \widehat{\theta}_k(0) = 0$, each ψ_k has integral 0. Let $k \in \mathbb{N}$ and $x \in [0, 2^{-k}]$. Note, for $j \geq 1$, we have $\mathrm{dist}_{\mathbb{R}}(x + j, [0, 2^{-k}]) = \mathrm{dist}_{\mathbb{R}}(x+j, 2^{-k}) = x + j - 2^{-k} \geq j - 2^{-k}$. For $j \leq -1$, we see $\mathrm{dist}_{\mathbb{R}}(x+j, [0, 2^{-k}]) = \mathrm{dist}_{\mathbb{R}}(x + j, 0) = |j| - x \geq |j| - 2^{-k}$. So,

$$
\begin{aligned}
|\psi_k(x)| \geq |\theta_k(x)| - \left|\sum_{j\neq 0} \theta_k(x+j)\right| &\geq c2^k - \sum_{j\neq 0} |\theta_k(x+j)| \\
&\geq c2^k - \sum_{j\neq 0} C_2 2^k \Big(1 + 2^k \mathrm{dist}_{\mathbb{R}}(x+j, [0, 2^{-k}])\Big)^{-2} \\
&\geq c2^k - C_2 2^k \sum_{j\neq 0} \Big(1 + 2^k(|j| - 2^{-k})\Big)^{-2} \\
&= c2^k - C_2 2^k \sum_{j\neq 0} (2^k|j|)^{-2} \;\geq\; 2^k\big[c - C_2 4^{1-k}\big]
\end{aligned}
$$

In particular, $|\psi_k| \geq \frac{c}{2} 2^k \chi_{[0,2^{-k}]}$ for all $k \geq K$, where K is the smallest integer with $K \geq \log(2C_2/c)(\log 4)^{-1} + 1$.

For each dyadic interval $I = [2^{-k}j, 2^{-k}(j+1)]$ with $k \geq K$, set $\varphi_I(x) = 2^{-k}\psi_k(x - 2^{-k}j)$. Each φ_I has 0 mean and is adapted to I with constants C_m' by Lemma 1.3. Further, $|\varphi_I(x)| = 2^{-k}|\psi_k(x - 2^{-k}j)| \geq \frac{c}{2}\chi_{[0,2^{-k}]}(x - 2^{-k}j) = a\chi_I(x)$, if $a = c/2$.

Let I be a dyadic interval with $|I| > 2^{-K}$, of which there are only finitely many. Choose a smooth function f_I so that $|f_I| \geq a\chi_I$. Let g_I be a smooth function, supported away from I, with $\int_{\mathbb{T}} g_I = 1$, and set $\varphi_I = f_I - (\int_{\mathbb{T}} f_I)g_I$. Then, $|\varphi_I| \geq a\chi_I$ and φ_I has mean 0. Do this for each remaining I, and choose C'' so that $\|\varphi_I\|_\infty, \|\varphi_I'\|_\infty \leq C''$ for all such I. Again, this is possible as there only finitely many. Set $C_m''' = (1 + 2^K)^m C''$. Then, for any $x \in \mathbb{T}$,

$$|\varphi_I(x)| \leq C''_m(1+2^K)^{-m} \leq C''_m\Big(1+\frac{1}{2|I|}\Big)^{-m} \leq C''_m\Big(1+\frac{\mathrm{dist}_{\mathbb{T}}(x,I)}{|I|}\Big)^{-m},$$

and

$$|\varphi'_I(x)| \leq C''_m(1+2^K)^{-m} \leq C''_m\frac{1}{|I|}\Big(1+\frac{1}{2|I|}\Big)^{-m} \leq C''_m\frac{1}{|I|}\Big(1+\frac{\mathrm{dist}_{\mathbb{T}}(x,I)}{|I|}\Big)^{-m}.$$

Hence, $\{\varphi_I\}_I$ is a 0-mean adapted family, with constants $\max(C'_m, C''_m)$, and $|\varphi_I| \geq a\chi_I$. □

The following is an important consequence of the definition, and the proof is the first of many which make use of a "geometric" argument and the adapted property.

Proposition 1.8. *For any adapted family φ_I, we have $\|\varphi_I\|_1 \lesssim |I|$, where the underlying constant does not depend on I.*

Proof. Fix I. If $|I| = 2^{-k}$, let $N = 2^{k-1}$ so that $\mathbb{T} = \bigcup\{I^m : -N+1 \leq m \leq N\}$ and this union is disjoint. Then,

$$\begin{aligned}
\|\varphi_I\|_1 = \int_{\mathbb{T}} |\varphi_I(x)|\,dx &= \sum_{m=-N+1}^{N} \int_{I^m} |\varphi_I(x)|\,dx \\
&\leq C_2 \sum_{m=-N+1}^{N} \int_{I^m} \Big(1+\frac{\mathrm{dist}_{\mathbb{T}}(x,I)}{|I|}\Big)^{-2} dx \\
&\leq C_2 \sum_{m=-N+1}^{N} \int_{I^m} \Big(1+\frac{\mathrm{dist}_{\mathbb{T}}(I^m,I)}{|I|}\Big)^{-2} dx.
\end{aligned}$$

Observe that $\mathrm{dist}_{\mathbb{T}}(I^m, I) = |I|(|m|-1)$ for $-N+1 \leq m \leq N$, $m \neq 0$. Thus,

$$
\begin{aligned}
\|\varphi_I\|_1 &\lesssim \sum_{m=-N+1}^{N} \int_{I^m} \Big(1 + \frac{\operatorname{dist}_{\mathbb{T}}(I^m, I)}{|I|}\Big)^{-2} dx \\
&= |I| + \sum_{-N+1 \le m \le N, m \neq 0} |I^m||m|^{-2} \\
&\le |I|\Big[1 + 2\sum_{m=1}^{N} \frac{1}{m^2}\Big] \le |I|\Big[1 + 2\sum_{m=1}^{\infty} \frac{1}{m^2}\Big] \lesssim |I|.
\end{aligned}
$$

□

Conceptually, we often think of functions which are adapted to an interval I as being "almost supported" in I. The following theorems give some rigid meaning to this.

Theorem 1.9. *Let $\varphi_I : \mathbb{T} \to \mathbb{C}$ be adapted to an interval I, with $|I| = 2^{-N}$. Then, we can write*

$$\varphi_I = \sum_{k=1}^{\infty} 2^{-10k} \varphi_I^k,$$

where φ_I^k are adapted to I uniformly in k. In addition, $\operatorname{supp}(\varphi_I^k) \subseteq 2^k I$ *for $1 \le k \le N$, and $\varphi_I^k = 0$ otherwise.*

Proof. Assume φ_I is adapted to I with constants C_m. Let $\psi : \mathbb{R} \to \mathbb{C}$ be smooth, supported in $[-1/2, 1/2]$, identically 1 on $[-1/4, 1/4]$, with $0 \le \psi \le 1$ and $|\psi'| \le 4$. For any interval J with center x_J, define $\psi_J(x) = \psi(\frac{x - x_J}{|J|})$. For each $0 \le k < N$, periodize the appropriate functions to create smooth functions $\psi_{2^k I}$ on $\mathbb{T}$ such that $0 \le \psi_{2^k I} \le 1$, $|\psi'_{2^k I}| \le 4/|I|$, $\operatorname{supp}(\psi_{2^k I}) \subseteq 2^k I$, and $\psi_{2^k I} = 1$ on $2^{k-1} I$.

We start by noting that

$$1 = \psi_I + (\psi_{2^2 I} - \psi_I) + \ldots + (\psi_{2^{N-1} I} - \psi_{2^{N-2} I}) + (1 - \psi_{2^{N-1} I}).$$

Therefore, if we define $\varphi_I^1 = 2^{10}\varphi_I\psi_I$, $\varphi_I^k = 2^{10k}\varphi_I(\psi_{2^kI} - \psi_{2^{k-1}I})$ for $1 < k < N$, $\varphi_I^N = 2^{10N}\varphi_I(1 - \psi_{2^{N-1}I})$, and $\varphi_I^k = 0$ for $k > N$, then

$$\varphi_I = \sum_{k=1}^{\infty} 2^{-10k}\varphi_I^k.$$

Further, $\operatorname{supp}(\varphi_I^k) \subseteq 2^kI$ by construction (for $k = N$, this is an empty statement).

Clearly, $|\varphi_I^1(x)| \le 2^{10}|\varphi_I(x)||\psi_I(x)| \le 2^{10}|\varphi_I(x)| \le 2^{10}C_m(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|})^{-m}$. Also, $|\varphi_I^{1\prime}(x)| \le 2^{10}|\varphi_I'(x)||\psi_I(x)| + 2^{10}|\varphi_I(x)||\psi_I'(x)| \le 2^{10}|\varphi_I'(x)| + 2^{10}\frac{4}{|I|}|\varphi_I(x)| \le 2^{10} \cdot 5C_m\frac{1}{|I|}(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|})^{-m}$.

Now, for each $1 < k < N$, $\psi_{2^kI} - \psi_{2^{k-1}I}$ is supported in $2^kI - 2^{k-2}I$. Also, $1 - \psi_{2^{N-1}I}$ is supported in $\mathbb{T} - 2^{N-2}I = 2^NI - 2^{N-2}I$. So, fix $1 < k \le N$ and let $x \in 2^kI - 2^{k-2}I$. Then, $2^{k-3} < \frac{\operatorname{dist}_{\mathbb{T}}(x,x_I)}{|I|} \le 2^{k-1}$, where x_I is the center of I. However, $\operatorname{dist}_{\mathbb{T}}(x, x_I) = \operatorname{dist}_{\mathbb{T}}(x, I) + |I|/2$, which gives $2^{k-3} - 1/2 < \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|} \le 2^{k-1} - 1/2$. Hence,

$$\begin{aligned}
|\varphi_I(x)| &\le C_{m+10}\Big(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|}\Big)^{-m-10} \le C_{m+10}(2^{k-3} + 1/2)^{-m-10} \\
&= C_{m+10}(2^{k-3} + 1/2)^{-10}(2^{k-3} + 1/2)^{-m} \\
&\le C_{m+10}\big(2^{30} \cdot 2^{-10k}\big)\big(4^m(2^{k-1} + 1/2)^{-m}\big) \\
&\le 4^{m+15}C_{m+10}2^{-10k}\Big(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|}\Big)^{-m}.
\end{aligned}$$

By precisely the same argument, $|\varphi_I'(x)| \le 4^{m+15}C_{m+10}2^{-10k}\frac{1}{|I|}(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|})^{-m}$. Thus, for all $x \in \mathbb{T}$,

$$\begin{aligned}
|\varphi_I^k(x)| &\le 2^{10k}|\varphi_I(x)| \le 4^{m+15}C_{m+10}\Big(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|}\Big)^{-m}, \\
|\varphi_I^{k\prime}(x)| &\le 2^{10k}\Big[|\varphi_I'(x)| + |\varphi_I(x)|\frac{8}{|I|}\Big] \le 9 \cdot 4^{m+15}C_{m+10}\frac{1}{|I|}\Big(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x,I)}{|I|}\Big)^{-m}.
\end{aligned}$$

In particular, φ_I^k is adapted to I with constants $9 \cdot 4^{m+15}C_{m+10}$ for all k. $\square$

Theorem 1.10. *Let $\varphi_I : \mathbb{T} \to \mathbb{C}$ be adapted to an interval I, $|I| = 2^{-N}$, with $\int_{\mathbb{T}} \varphi_I \, dm = 0$. Then, we can write*

$$\varphi_I = \sum_{k=1}^{\infty} 2^{-10k} \varphi_I^k,$$

where φ_I^k are adapted to I uniformly in k and $\int \varphi_I^k \, dm = 0$. In addition, $\operatorname{supp}(\varphi_I^k) \subseteq 2^k I$ *for $1 \leq k \leq N$, and $\varphi_I^k = 0$ otherwise.*

Proof. Using Theorem 1.9, write $\varphi_I = \sum 2^{-10k} \varphi_I^k$, where $\operatorname{supp}(\varphi_I^k) \subseteq 2^k I$ for $1 \leq k \leq N$ and $\varphi_I^k = 0$ otherwise. Further, φ_I^k are adapted to I with uniform constants.

Choose a smooth function $\psi : \mathbb{T} \to \mathbb{C}$ so that $0 \leq \psi \leq 2/|I|$, $|\psi'| \leq 8/|I|^2$, $\int \psi \, dm = 1$, and $\operatorname{supp}(\psi) \subseteq I$. Set $\varphi_{0,I}^k = \varphi_I^k - (\int \varphi_I^k \, dm)\psi$. Then, each $\varphi_{0,I}^k$ has integral 0, and is still supported in $2^k I$. Further,

$$\sum_{k=1}^{\infty} 2^{-10k} \varphi_{0,I}^k = \sum_{k=1}^{\infty} 2^{-10k} \varphi_I^k - \psi\Big(\int_{\mathbb{T}} \sum_{k=1}^{\infty} 2^{-10k} \varphi_I^k \, dm\Big) = \varphi_I - \psi\Big(\int_{\mathbb{T}} \varphi_I \, dm\Big) = \varphi_I.$$

As φ_I^k are uniformly adapted to I, we see by Proposition 1.8 that $\|\varphi_I^k\|_1 \lesssim |I|$. So, for $x \in I$,

$$\Big|\Big(\int \varphi_I^k \, dm\Big)\psi(x)\Big| \lesssim 1 = \Big(1 + \frac{\operatorname{dist}(x, I)}{|I|}\Big)^{-m},$$
$$\Big|\Big(\int \varphi_I^k \, dm\Big)\psi'(x)\Big| \lesssim \frac{1}{|I|} = \frac{1}{|I|}\Big(1 + \frac{\operatorname{dist}(x, I)}{|I|}\Big)^{-m}.$$

Of course, for $x \notin I$, these quantities are 0. It follows that $\varphi_{0,I}^k$ are uniformly adapted to I. $\square$

1.5 Interpolation Theorems

Let (X, ρ) be a measure space and $(B, \|\cdot\|_B)$ a (complex) Banach space and its associated norm. Consider functions $f : (X, \rho) \to B$ which take values in this Banach space. Let $\mathcal{M}(X, B)$ be the set of such functions such that the map $x \mapsto \|f(x)\|_B$ is measurable.

For $0 < p < \infty$ and $f \in \mathcal{M}(X, B)$, define

$$\|f\|_{p,B} = \Big(\int_X \|f(x)\|_B^p \, \rho(dx) \Big)^{1/p},$$

and $\|f\|_{\infty,B} = \operatorname{ess\,sup}_X \|f(x)\|_B$. Let $L_B^p(X)$ be the set of functions for which these quantities are finite. It is easily established that $L_B^p(X)$ are Banach spaces, as usual, for $1 \le p \le \infty$. Let

$$\|f\|_{p,\infty,B} = \sup_{\lambda > 0} \lambda \, \rho\{x \in X : \|f(x)\|_B > \lambda\}^{1/p},$$

and define $L_B^{p,\infty}(X)$ accordingly.

The principal reason for considering such spaces is to attain interpolation results for operators T which take $\mathcal{M}(X, B)$ to $\mathcal{M}(X, B)$. We say an operator is sublinear if $\|T(f+g)(x)\|_B \le \|Tf(x)\|_B + \|Tg(x)\|_B$ and $\|T(\alpha f)(x)\|_B = |\alpha| \|Tf(x)\|_B$ for all scalars $\alpha \in \mathbb{C}$ and almost every $x \in X$. Consider the following [8].

Theorem 1.11. *Let T be a sublinear operator on $\mathcal{M}(X, B)$. Suppose that for some $0 < p_0 < p_1 \le \infty$, $T : L_B^{p_j}(X) \to L_B^{p_j,\infty}(X)$ for $j = 0, 1$ (where $L_B^{\infty,\infty} = L_B^\infty$). Then, for every $p_0 < p < p_1$, $T : L_B^p(X) \to L_B^p(X)$.*

Proof. Fix p and f. First, suppose $p_1 < \infty$. For each $t > 0$, let $f^t = f$ when $\|f\|_B > t$ and 0 otherwise. Similarly, let $f_t = f$ when $\|f\|_B \le t$ and 0 otherwise, so that $f = f_t + f^t$.

Note, $\|Tf(x)\|_B \le \|Tf_t(x)\|_B + \|Tf^t(x)\|_B$, and by hypothesis,

$$\rho\{x : \|Tf(x)\|_B > t\} \leq \rho\{\|Tf^t\|_B > t/2\} + \rho\{\|Tf_t\|_B > t/2\}$$
$$\lesssim (t/2)^{-p_0}\|f^t\|_{p_0,B}^{p_0} + (t/2)^{-p_1}\|f_t\|_{p_1,B}^{p_1},$$

where the underlying constants are the operator norms of T. So,

$$\begin{aligned}
\|Tf\|_{p,B}^p &= \int_X \|Tf\|_B^p \, d\rho = p\int_0^\infty t^{p-1}\rho\{\|Tf\|_B > t\}\, dt \\
&\lesssim \int_0^\infty t^{p-p_0-1}\|f^t\|_{p_0,B}^{p_0} + t^{p-p_1-1}\|f_t\|_{p_1,B}^{p_1}\, dt \\
&= \int_0^\infty t^{p-p_0-1}\int_{\{\|f\|_B > t\}} \|f\|_B^{p_0}\, d\rho\, dt + \int_0^\infty t^{p-p_1-1}\int_{\{\|f\|_B \leq t\}} \|f\|_B^{p_1}\, d\rho\, dt \\
&= \int_X \|f\|_B^{p_0}\int_0^{\|f\|_B} t^{p-p_0-1}\, dt\, d\rho + \int_X \|f\|_B^{p_1}\int_{\|f\|_B}^\infty t^{p-p_1-1}\, dt\, d\rho \\
&= \frac{1}{p-p_0}\int_X \|f\|_B^p\, d\rho + \frac{1}{p_1-p}\int_X \|f\|_B^p\, d\rho \lesssim \|f\|_{p,B}^p.
\end{aligned}$$

Now, suppose $p_1 = \infty$. Let C be the operator norm of $T : L_B^\infty \to L_B^\infty$. For each $t > 0$, set $f_t = f$ for $\|f\|_B \leq t/(2C)$ and 0 otherwise. Define f^t accordingly so that $f = f_t + f^t$. Note, $\|Tf_t(x)\|_B \leq \|Tf_t\|_{\infty,B} \leq C\|f_t\|_{\infty,B} \leq t/2$ for almost every $x \in X$. Thus, $\rho\{x : \|Tf_t(x)\|_B > t/2\} = 0$. Hence,

$$\begin{aligned}
\|Tf\|_{p,B}^p &= \int_X \|Tf\|_B^p\, d\rho = p\int_0^\infty t^{p-1}\rho\{\|Tf\|_B > t\}\, dt \\
&\lesssim \int_0^\infty t^{p-p_0-1}\|f^t\|_{p_0,B}^{p_0}\, dt \\
&= \int_0^\infty t^{p-p_0-1}\int_{\{\|f\|_B > t/(2C)\}} \|f\|_B^{p_0}\, d\rho\, dt \\
&= \int_X \|f\|_B^{p_0}\int_0^{2C\|f\|_B} t^{p-p_0-1}\, dt\, d\rho \\
&= \frac{(2C)^{p-p_0}}{p-p_0}\int_X \|f\|_B^p\, d\rho \lesssim \|f\|_{p,B}^p.
\end{aligned}$$

□

The preceding theorem is a generalization of the classical Marcinkiewicz interpolation theorem [25, 29]. Indeed, the proof is nearly identical. To recover the classical version, we need only take the Banach space B to be $\mathbb{C}$ with norm $|\cdot|$. Like the Marcinkiewicz interpolation theorem, we can actually prove a version where $T : L^{p_j}_B \to L^{q_j,\infty}_B$, for $j = 0, 1$, implies $T : L^p_B \to L^q_B$, with the standard relationships between p, p_0, p_1 and q, q_0, q_1. However, the proof presented here is slightly neater, and is all we will need.

Chapter 2
Maximal Operators

Given an adapted family φ_I and a function $f : \mathbb{T} \to \mathbb{C}$, we will be interested in "averages" of f with respect to the family. In particular, given the sequence

$$\left\{ \frac{1}{|I|} |\langle \varphi_I, f \rangle| \chi_I(x) \right\}_I,$$

the associated ℓ^2 and ℓ^∞-norms will be useful quantities. The ℓ^∞-norm is examined in this chapter. The ℓ^2-norm is the principal subject of Chapter 3. Let

$$M' f(x) = \sup_I \frac{1}{|I|} |\langle \varphi_I, f \rangle| \chi_I(x).$$

Instead of studying this operator directly, it will be more useful to study a different, but related operator; one which is independent of any adapted family.

2.1 Hardy-Littlewood Maximal Function

Definition. For $f : \mathbb{T} \to \mathbb{C}$, define the Hardy-Littlewood maximal function [10] by

$$M f(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)| \, dy,$$

where the supremum is taken over all intervals in $\mathbb{T}$ containing x. Similarly, define the dyadic maximal function $M_D f(x)$ where the supremum is instead taken over all dyadic intervals containing x.

We will not be interested in proving results for M_D, per se, but it will prove a useful tool in this and other chapters.

Proposition 2.1. *For any complex-valued functions f_k, f, g on $\mathbb{T}$ and any scalars α,*

1. $M(f+g) \le Mf + Mg$ *and* $M(\alpha f) = |\alpha| Mf$,
2. $|f| \le |g|$ *a.e. implies* $Mf \le Mg$ *pointwise,*
3. $|f_k| \uparrow |f|$ *a.e. implies* $Mf_k \uparrow Mf$ *pointwise.*

The same is true for M_D.

Proof. The proofs for M and M_D are essentially identical, and we handle only the M case.

(1) Fix $x \in \mathbb{T}$ and an interval I containing x. Then, it is clear that $|I|^{-1} \int_I |f(y) + g(y)|\, dy \le |I|^{-1} \int_I |f(y)|\, dy + |I|^{-1} \int_I |g(y)|\, dy \le Mf(x) + Mg(x)$. As I is arbitrary, $M(f+g)(x) \le Mf(x) + Mg(x)$. Also, $M(\alpha f)(x) = \sup |I|^{-1} \int_I |\alpha f(y)|\, dy = |\alpha| \sup |I|^{-1} \int_I |f(y)|\, dy = |\alpha| Mf(x)$.

(2) Fix $x \in \mathbb{T}$ and an interval I containing x. Then, $|I|^{-1} \int_I |f(y)|\, dy \le |I|^{-1} \int_I |g(y)|\, dy \le Mg(x)$, which implies $Mf(x) \le Mg(x)$.

(3) From statement 2 above, $Mf_1 \le Mf_2 \le \ldots \le Mf$. Fix $x \in \mathbb{T}$ and $\epsilon > 0$. There exists an interval I containing x so that $Mf(x) \le |I|^{-1} \int_I |f(y)|\, dy + \epsilon/2$. By the monotone convergence theorem, $\int_I |f_k(y)|\, dy \uparrow \int_I |f(y)|\, dy$. So, choose N such that $k \ge N$ implies $|I|^{-1} \int_I |f_k(y)|\, dy > |I|^{-1} \int_I |f(y)|\, dy - \epsilon/2$. Then, for all $k \ge N$, it follows $Mf(x) \le |I|^{-1} \int_I |f(y)|\, dy + \epsilon/2 \le |I|^{-1} \int_I |f_k(y)|\, dy + \epsilon \le Mf_k(x) + \epsilon$. As $\epsilon > 0$ is arbitrary, $Mf_k(x) \uparrow Mf(x)$. $\square$

Proposition 2.2. *For any $f : \mathbb{T} \to \mathbb{C}$, we have $M'f \lesssim Mf$ pointwise, where the underlying constant is independent of f.*

Proof. Let φ_I be an adapted family and $f : \mathbb{T} \to \mathbb{C}$. By Theorem 1.9, write

$$\varphi_I = \sum_{k=1}^{\infty} 2^{-10k} \varphi_I^k,$$

for each I, where φ_I^k are uniformly adapted to I. In particular, $\|\varphi_I^k\|_\infty \lesssim 1$ uniformly in I and k. Further, $\mathrm{supp}(\varphi_I^k) \subseteq 2^k I$ when k is small enough and identically 0 otherwise.

Fix I, and suppose $|I| = 2^{-n}$. Let $x \in I$. Then,

$$\begin{aligned}
\frac{1}{|I|}|\langle \varphi_I, f\rangle|\chi_I(x) &\leq \frac{1}{|I|}\sum_{k=1}^{n} 2^{-10k}\int_{\mathbb{T}} |f(x)||\varphi_I^k(x)|\,dx = \frac{1}{|I|}\sum_{k=1}^{n}\int_{2^kI}|f(x)||\varphi_I^k(x)|\,dx \\
&\lesssim \frac{1}{|I|}\sum_{k=1}^{n} 2^{-10k}\int_{2^kI}|f(x)|\,dx = \sum_{k=1}^{n} 2^{-9k}\frac{1}{|2^kI|}\int_{2^kI}|f(x)|\,dx \\
&\leq \sum_{k=1}^{n} 2^{-9k} Mf(x) \leq \sum_{k=1}^{\infty} 2^{-9k} Mf(x) \lesssim Mf(x).
\end{aligned}$$

Of course, if $x \notin I$, this holds trivially. As I is arbitrary, take the supremum to see $M'f(x) \lesssim Mf(x)$. □

In light of this, any boundedness property of M will hold automatically for M'. Therefore, we restrict our attention to M for the remainder of the chapter.

For any interval $I \subseteq \mathbb{T}$, denote by $I^* = 3I$, if $|I| \leq 1/3$, and $I^* = \mathbb{T}$ if $|I| > 1/3$. Thus, $I \subseteq I^*$ and $|I^*| \leq 3|I|$.

Claim 2.3. *Let A, B be intervals in $\mathbb{T}$. If $A \cap B$ is non-empty and $|B| \leq |A|$, then $B \subseteq A^*$.*

Proof. Suppose A, B have centers x_A, x_B. Pick $z \in A \cap B$. Then, for $x \in B$,

$$\begin{aligned}
\mathrm{dist}(x, x_A) &\leq \mathrm{dist}(x, x_B) + \mathrm{dist}(x_B, z) + \mathrm{dist}(z, x_A) \\
&\leq |B|/2 + |B|/2 + |A|/2 \leq 3|A|/2.
\end{aligned}$$

Namely, $x \in A^*$ and $B \subseteq A^*$. □

Of course, $\operatorname{dist}(\cdot,\cdot)$ refers to $\operatorname{dist}_{\mathbb{T}}(\cdot,\cdot)$. As we will be exclusively on $\mathbb{T}$ from now on, we no longer make this distinction.

Claim 2.4. *Let A, B be intervals in $\mathbb{T}$. If $A \cap B$ and $A - B^*$ are both nonempty, then $B \subseteq A^*$.*

Proof. Suppose A, B have centers x_A, x_B. Let $u \in A \cap B$ and $v \in A - B^*$. That is, $\operatorname{dist}(u, x_A) \leq |A|/2$ and $\operatorname{dist}(u, x_B) \leq |B|/2$. Also, $\operatorname{dist}(v, x_A) \leq |A|/2$, but $\operatorname{dist}(v, x_B) > 3|B|/2$. Then,

$$\begin{aligned} 3|B|/2 < \operatorname{dist}(v, x_B) &\leq \operatorname{dist}(v, x_A) + \operatorname{dist}(x_A, u) + \operatorname{dist}(u, x_B) \\ &\leq |A|/2 + |A|/2 + |B|/2, \end{aligned}$$

which implies $|B| < |A|$. It now follows by Claim 2.3 that $B \subseteq A^*$. $\square$

The following is a decomposition lemma similar to that of Calderón and Zygmund [3], of which we will ultimately prove several different versions.

Lemma 2.5. *Let $f : \mathbb{T} \to \mathbb{C}$ and $\alpha > 0$ so that $\{Mf > \alpha\}$ is non-empty. Then, there exists a sequence of disjoint intervals I_j such that $\{Mf > \alpha\} \subseteq \bigcup_j I_j^*$ and*

$$\frac{\alpha}{4} \leq \frac{1}{|I_j|} \int_{I_j} |f(x)|\, dx \text{ for all } I_j.$$

Proof. Let $\Omega = \{M_D f > \alpha/4\}$. Assume Ω is non-empty. This will be justified shortly. Let $\mathcal{D}$ be the countable collection of all dyadic intervals I such that $\frac{1}{|I|} \int_I |f(y)|\, dy > \alpha/4$. By construction, $\Omega = \bigcup_{\mathcal{D}} I$. We say a dyadic interval $I \in \mathcal{D}$ is maximal if for every $I' \in \mathcal{D}$, we have either $I' \subseteq I$ or I, I' are disjoint. Clearly, every $I \in \mathcal{D}$ is contained in a maximal interval. Let $I_1, I_2, \ldots$ be the maximal intervals of $\mathcal{D}$, which are necessarily disjoint. Further, it is clear that

$$\Omega = \bigcup_{\mathcal{D}} I = \bigcup_{\mathbb{N}} I_j.$$

Let $x \in \{Mf > \alpha\}$. By definition, there is an interval J containing x so that $\frac{1}{|J|}\int_J |f|\,dm > \alpha$. Write $J = [a, a+|J|]$. Choose $k \in \mathbb{N}$ so that $2^{-k-1} \leq |J| < 2^{-k}$, and pick an integer j so that $(j-1)2^{-k} \leq a < j2^{-k}$. Then, $a + |J| < (j+1)2^{-k}$. Let $I = [2^{-k}(j-1), 2^{-k}j]$ and $I' = [2^{-k}j, 2^{-k}(j+1)]$, which are both dyadic and $J \subseteq I \cup I'$. It follows that either

$$\int_{I\cap J} |f(x)|\,dx > \alpha|J|/2 \quad \text{or} \quad \int_{I'\cap J} |f(x)|\,dx > \alpha|J|/2.$$

Without loss of generality, assume it is the first. But, $|J| \geq 2^{-k-1} = |I|/2$. Thus, $\int_I |f(x)|\,dx > \alpha|I|/4$, or $I \in \mathcal{D}$ (it is now clear that Ω is non-empty). So, $I \subseteq I_j$ for some j. As $I \cap J$ is non-empty and $|J| \leq |I|$, we have by Claim 2.3 that $x \in J \subseteq I^* \subseteq I_j^*$. As x is arbitrary, $\{Mf > \alpha\} \subseteq \bigcup_j I_j^*$. □

Theorem 2.6. $M : L^1 \to L^{1,\infty}$.

Proof. Fix $\alpha > 0$ and set $E = \{Mf > \alpha\}$. If E is empty, the $|E| \leq \|f\|_1/\alpha$ trivially. Assume it is not empty, and apply Lemma 2.5 to find disjoint intervals I_j. Then,

$$|E| \leq \sum_j |I_j^*| \leq 3\sum_j |I_j| \lesssim \frac{1}{\alpha}\sum_j \int_{I_j} |f(x)|\,dx = \frac{1}{\alpha}\int_{\bigcup_j I_j} |f(x)|\,dx \leq \frac{1}{\alpha}\|f\|_1.$$

As $\alpha > 0$ is arbitrary, this completes the proof. □

Corollary 2.7. $M : L^p \to L^p$ *for all* $1 < p \leq \infty$.

Proof. As M is sublinear, it suffices by the Marcinkiewicz interpolation theorem to show $M : L^\infty \to L^\infty$. But, for any $x \in \mathbb{T}$ and any interval I containing x,

$$\frac{1}{|I|}\int_I |f(y)|\,dy \leq \|f\|_\infty,$$

which implies $\|Mf\|_\infty \leq \|f\|_\infty$. $\square$

Corollary 2.8. *For $f \in L^1(\mathbb{T})$, $|f| \leq M_D f \leq Mf$ a.e..*

Proof. The fact that $M_D f \leq Mf$ pointwise is clear, as the supremum in M is taken over a larger class of sets.

For each $x \in \mathbb{T}$, let $I_k(x)$ be the dyadic interval containing x with $|I| = 2^{-k}$. Define

$$V_f(x) = \limsup_{k\to\infty} \frac{1}{|I_k(x)|}\int_{I_k(x)} |f(y) - f(x)|\,dy.$$

Let $\epsilon > 0$. As the continuous functions are dense in $L^1(\mathbb{T})$, choose g continuous so that $\|h\|_1 < \epsilon$ where $f = g+h$. Define V_g and V_h accordingly, and note $V_f \leq V_g + V_h$.

Fix $x \in \mathbb{T}$ and let $\delta > 0$. As g is continuous at x, there is some r so that $|x-y| < r$ implies $|g(x) - g(y)| < \delta$. Then, for all $k > -\log_2 r$, we see

$$\frac{1}{|I_k(x)|}\int_{I_k(x)} |g(y) - g(x)|\,dy < \delta.$$

That is, $V_g(x) \leq \delta$. As δ and x are arbitrary, $V_g = 0$. So, $V_f \leq V_h$.

On the other hand, we clearly have

$$\begin{aligned} V_h(x) &= \limsup_k \frac{1}{|I_k(x)|}\int_{I_k(x)} |h(y) - h(x)|\,dy \\ &\leq \limsup_k \frac{1}{|I_k(x)|}\int_{I_k(x)} |h(y)|\,dy + |h(x)| \leq M_D h(x) + |h(x)|. \end{aligned}$$

Thus, for all $t > 0$,

$$
\begin{aligned}
|\{x \in \mathbb{T} : V_f(x) > t\} &\leq |\{V_h > t\}| \\
&\leq |\{M_D h > t/2\}| + |\{|h| > t/2\}| \\
&\leq \frac{2}{t}\|M\|_{L^1 \to L^{1,\infty}}\|h\|_1 + \frac{2}{t}\|h\|_1 \\
&\leq \frac{2\epsilon}{t}(1 + \|M\|_{L^1 \to L^{1,\infty}}).
\end{aligned}
$$

As $\epsilon > 0$ is arbitrary, $|\{V_f > t\}| = 0$. As t is arbitrary, $V_f = 0$ a.e.. Namely, $f(x) = \lim_k |I_k(x)|^{-1} \int_{I_k(x)} |f(y)|\, dy \leq M_D f(x)$ a.e.. $\square$

2.2 Fefferman-Stein Inequalities

Our goal in this section will be to prove the classical Fefferman-Stein inequalities [7] below.

Theorem. *For any sequence $f_1, f_2, \ldots$ of complex-valued functions on $\mathbb{T}$ and any $1 < p, r < \infty$*

$$
\begin{aligned}
\Big\| \Big(\sum_{k=1}^{\infty} |Mf_k|^r \Big)^{1/r} \Big\|_p &\lesssim \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \Big\|_p, \\
\Big\| \Big(\sum_{k=1}^{\infty} |Mf_k|^r \Big)^{1/r} \Big\|_{1,\infty} &\lesssim \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \Big\|_1,
\end{aligned}
$$

where the underlying constants depend only on p and r.

Note the similarities between these results and what we know about M. In both cases, we have $L^1 \to L^{1,\infty}$ and $L^p \to L^p$ $(1 < p < \infty)$ results. But, here, there can be no L^∞ estimate. Indeed, fix $1 < r < \infty$ and set $f_k = \chi_{[2^{-k-1}, 2^{-k})}$. Then, $(\sum |f_k|^r)^{1/r} = \chi_{(0,1/2)}$, which has L^∞-norm 1. But, if $x \in [0, 2^{-N}]$, then $x \in [0, 2^{-k}]$ for any $k \leq N$. So, $Mf_k(x) \geq |[0, 2^{-k}]|^{-1} \int_{[0,2^{-k}]} f_k(y)\, dy = 1/2$.

Namely, $(\sum |Mf_k(x)|^r)^{1/r} \geq N^{1/r}/2$ for every $x \in [0, 2^{-N}]$. As N is arbitrary, $\|(\sum |Mf_k|^r)^{1/r}\|_\infty = \infty$.

One can also see that no $r = 1$ estimate could exist. Fix a positive integer N and set $f_k = \chi_{[(k-1)/N, k/N)}$ for $k \leq N$ and $f_k = 0$ for $k > N$. Then, $\sum |f_k| = 1$, which has L^p-norm 1 for every $1 \leq p \leq \infty$. On the other hand, fix $x \in \mathbb{T}$. Choose $1 \leq j \leq N$ so that $x \in [\frac{j-1}{N}, \frac{j}{N})$. Fix $1 \leq k \leq N$ and denote $r = \frac{|k-j|}{N} + \frac{1}{N}$. Then, there is an interval I, with $|I| = r$, containing $[\frac{k-1}{N}, \frac{k}{N}]$ and $[\frac{j-1}{N}, \frac{j}{N}]$, thus x. So,

$$Mf_k(x) \geq \frac{1}{|I|} \int_I f_k(y)\, dy = \frac{1}{rN} = \frac{1}{|k-j|+1}.$$

Hence, $\sum |Mf_k(x)| \geq \sum_{k=1}^N \frac{1}{|k-j|+1} \geq \sum_{k=1}^N \frac{1}{k+1} \geq \log N - 1$. This holds for all x, so $\| \sum |Mf_k| \|_p \geq \log N - 1$. As N is arbitrary, no $r = 1$ estimate could exist. These two counterexamples are taken from Stein [32].

Finally, we note that the $r = \infty$ case also holds (even for $p = \infty$), almost trivially. One only needs to note that $\sup_k Mf_k \leq M(\sup_k |f_k|)$ pointwise, and apply the L^p theory of M.

Lemma 2.9. *Let $f \in L^1(\mathbb{T})$ and $\alpha > \|f\|_1$ a constant. Then, there exists a sequence of disjoint dyadic intervals I_j such that, if $\Omega = \bigcup_j I_j$, then $|f| \leq \alpha$ a.e. on Ω^c and*

$$|\Omega| = \sum_{j=1}^{\infty} |I_j| \leq \frac{1}{\alpha}\|f\|_1,$$

$$\frac{1}{|I_j|} \int_{I_j} |f(x)|\, dx \leq 2\alpha \text{ for all } I_j.$$

Proof. Define $\Omega = \{M_D f > \alpha\}$. As $|f| \leq M_D f$ a.e., we see immediately that $|f| \leq M_D f \leq \alpha$ a.e. on Ω^c. If Ω is empty, then $|f| \leq \alpha$ everywhere. Thus,

$|I|^{-1}\int_I |f(y)|\,dy \leq \alpha$ for any interval I. Simply choose a dyadic interval I_1 so that $|I_1| \leq \|f\|_1/\alpha$, and let I_j be empty for $j > 1$. Then, $|\Omega| \leq \|f\|_1/\alpha$, and all conditions are satisfied.

Now, assume Ω is not empty. Let $\mathcal{D}$ be the countable collection of all dyadic intervals I such that $\frac{1}{|I|}\int_I |f(y)|\,dy > \alpha$. By construction, $\Omega = \bigcup_{\mathcal{D}} I$. We say a dyadic interval $I \in \mathcal{D}$ is maximal if for every $I' \in \mathcal{D}$, we have either $I' \subseteq I$ or I, I' are disjoint. Clearly, every $I \in \mathcal{D}$ is contained in a maximal interval. Let $I_1, I_2, \ldots$ be the maximal intervals of $\mathcal{D}$, which are necessarily disjoint. Further, it is clear that

$$\Omega = \bigcup_{\mathcal{D}} I = \bigcup_{\mathbb{N}} I_k.$$

As each $I_k \in \mathcal{D}$, we have $\alpha|I_k| < \int_{I_k} |f(y)|\,dy$. As the I_k are disjoint, simply sum over k to see $\alpha|\Omega| \leq \int_\Omega |f(y)|\,dy \leq \|f\|_1$. On the other hand, if $|I_k| < 1/2$, then there is some dyadic interval I_k' which contains I_k and satisfies $|I_k'| = 2|I_k|$. But, $I_k' \notin \mathcal{D}$, because otherwise I_k could not be maximal. Thus, $\alpha|I_k'| \geq \int_{I_k'} |f(y)|\,dy$, which implies $\int_{I_k} |f(y)|\,dy \leq \int_{I_k'} |f(y)|\,dy \leq \alpha|I_k'| = 2\alpha|I_k|$. Similarly, if $|I_k| = 1/2$, then $\int_{I_k} |f(y)|\,dy \leq \|f\|_1 < \alpha = 2\alpha|I_k|$. $\square$

Lemma 2.10. *For any sequence $f_1, f_2, \ldots$ on $\mathbb{T}$ and $1 < r < \infty$*

$$\left\|\Big(\sum_{k=1}^{\infty} |Mf_k|^r\Big)^{1/r}\right\|_r \lesssim \left\|\Big(\sum_{k=1}^{\infty} |f_k|^r\Big)^{1/r}\right\|_r,$$

where the underlying constants depend only on r.

Proof. Simply note that

$$\left\| \Big(\sum_{k=1}^{\infty} |Mf_k|^r \Big)^{1/r} \right\|_r^r = \int_{\mathbb{T}} \Big(\sum_{k=1}^{\infty} |Mf_k(x)|^r \Big)\, dx = \sum_{k=1}^{\infty} \int_{\mathbb{T}} |Mf_k(x)|^r\, dx$$
$$\leq \|M\|_{L^r \to L^r}^r \sum_{k=1}^{\infty} \int_{\mathbb{T}} |f_k(x)|^r\, dx$$
$$= \|M\|_{L^r \to L^r}^r \left\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \right\|_r^r .$$

□

Theorem 2.11. *For any sequence $f_1, f_2, \ldots$ on $\mathbb{T}$ and $1 < r < \infty$*

$$\left\| \Big(\sum_{k=1}^{\infty} |Mf_k|^r \Big)^{1/r} \right\|_{1,\infty} \lesssim \left\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \right\|_1 ,$$

where the underlying constants depend only on r.

Proof. Denote $F(x) = (\sum_{k=1}^{\infty} |f_k(x)|^r)^{1/r} \geq 0$. If F is not in L^1, then there is nothing to prove. So, assume $F \in L^1(\mathbb{T})$. Let $\alpha > \|F\|_1$. Then, by applying Lemma 2.9 to F and α, find disjoint intervals I_j, $\Omega = \bigcup I_j$, satisfying

$$(a) \quad |\Omega| = \sum_{j=1}^{\infty} |I_j| \leq \frac{1}{\alpha} \|F\|_1,$$
$$(b) \quad F \leq \alpha \text{ on } \Omega^c,$$
$$(c) \quad \frac{1}{|I_j|} \int_{I_j} F(y)\, dy \leq 2\alpha \text{ for each } I_j.$$

Decompose each f_k into $f_k = f_k' + f_k''$ where $f_k' = f_k \chi_{\Omega^c}$ and $f_k'' = f_k \chi_{\Omega}$. Denote $F' = (\sum |f_k'|^r)^{1/r}$.

As $f_k' \leq f_k$ pointwise, it is clear that $F' \leq F$ pointwise. On the other hand, F' is 0 on Ω. So, by (b) above, we see $F' \leq \alpha$. Thus,

$$\|F'\|_r^r = \int_{\mathbb{T}} |F'(x)|^r\, dx \leq \alpha^{r-1} \|F'\|_1 \leq \alpha^{r-1} \|F\|_1.$$

Applying Lemma 2.10, we have

$$\Big\|\Big(\sum_{k=1}^{\infty}|Mf_k'|^r\Big)^{1/r}\Big\|_r^r \lesssim \Big\|\Big(\sum_{k=1}^{\infty}|f_k'|^r\Big)^{1/r}\Big\|_r^r = \|F'\|_r^r \leq \alpha^{r-1}\|F\|_1.$$

An application of Chebyshev's inequality yields

$$\Big|\Big\{\Big(\sum_{k=1}^{\infty}|Mf_k'|^r\Big)^{1/r} > \alpha/2\Big\}\Big| \lesssim \frac{1}{\alpha^r}\Big\|\Big(\sum_{k=1}^{\infty}|Mf_k'|^r\Big)^{1/r}\Big\|_r^r \lesssim \frac{1}{\alpha}\|F\|_1.$$

On the other hand, define functions g_k by

$$g_k(x) = \begin{cases} \frac{1}{|I_j|}\int_{I_j}|f_k(y)|\,dy, & \text{if } x \in I_j, \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

As the I_j are disjoint, this is well-defined a.e.. Let $G(x) = (\sum |g_k|^r)^{1/r}$, which is supported on Ω.

Fix $x \in \Omega$. Then, x is in some I_j. By the generalized Minkowski inequality (see Lieb and Loss [20] or Rudin [28]) and (c) above, we have

$$\begin{aligned} G(x) &= \Big(\sum_{k=1}^{\infty}\Big[\frac{1}{|I_j|}\int_{I_j}|f_k(y)|\,dy\Big]^r\Big)^{1/r} \leq \frac{1}{|I_j|}\int_{I_j}\Big(\sum_{k=1}^{\infty}|f_k(y)|^r\Big)^{1/r}dy \\ &= \frac{1}{|I_j|}\int_{I_j}F(y)\,dy \leq 2\alpha. \end{aligned}$$

Hence, as G is supported in Ω and bounded by 2α, we see $\|G\|_r^r \lesssim \alpha^r|\Omega| \leq \alpha^{r-1}\|F\|_1$. Precisely as was done above, apply Lemma 2.10 and Chebyshev to see

$$\Big|\Big\{\Big(\sum_{k=1}^{\infty}|Mg_k|^r\Big)^{1/r} > \alpha/6\Big\}\Big| \lesssim \frac{1}{\alpha^r}\Big\|\Big(\sum_{k=1}^{\infty}|Mg_k|^r\Big)^{1/r}\Big\|_r^r \lesssim \frac{1}{\alpha}\|F\|_1.$$

Now, we would now like to establish some relationship between Mg_k and Mf_k''. First, note that for any I_j,

$$\int_{I_j}|g_k(x)|\,dx = \int_{I_j}\Big(\frac{1}{|I_j|}\int_{I_j}|f_k(y)|\,dy\Big)dx = \int_{I_j}|f_k(y)|\,dy = \int_{I_j}|f_k''(y)|\,dy.$$

Set $\Omega^* = \bigcup I_j^*$. By (a),

$$|\Omega^*| \leq \sum_j |I_j^*| \leq 3 \sum_j |I_j| \lesssim \frac{1}{\alpha}\|F\|_1.$$

Fix $x \notin \Omega^*$ and I an interval containing x. As each f_k'' is supported on $\Omega = \bigcup I_j$, we see

$$\frac{1}{|I|}\int_I |f_k''(y)|\,dy = \frac{1}{|I|}\sum_{j\in\mathbb{N}}\int_{I\cap I_j}|f_k''(y)|\,dy = \frac{1}{|I|}\sum_{j\in J}\int_{I\cap I_j}|f_k''(y)|\,dy,$$

where $J = \{j : I_j \cap I \neq \emptyset\}$. But, for $j \in J$, we have $I_j \cap I \neq \emptyset$ and $x \in I - \Omega^* \subseteq I - I_j^*$. By Claim 2.4, this implies $I_j \subseteq I^*$. So,

$$\begin{aligned}
\frac{1}{|I|}\int_I |f_k''(y)|\,dy &= \frac{1}{|I|}\sum_J \int_{I\cap I_j}|f_k''(y)|\,dy \leq \frac{1}{|I|}\sum_J\int_{I_j}|f_k''(y)|\,dy \\
&= \frac{1}{|I|}\sum_J\int_{I_j}|g_k(y)|\,dy \leq \frac{1}{|I|}\int_{I^*}|g_k(y)|\,dy \\
&\leq \frac{3}{|I^*|}\int_{I^*}|g_k(y)|\,dy \leq 3Mg_k(x).
\end{aligned}$$

As I is arbitrary, $Mf_k''(x) \leq 3Mg_k(x)$. As $x \notin \Omega^*$ is arbitrary, this holds on $\mathbb{T}-\Omega^*$. Hence, $(\sum |Mf_k''|^r)^{1/r} \leq 3(\sum |Mg_k|^r)^{1/r}$ on $\mathbb{T}-\Omega^*$, and

$$\begin{aligned}
&\left|\left\{x \in \mathbb{T}-\Omega^* : \Big(\sum_{k=1}^\infty |Mf_k''(x)|^r\Big)^{1/r} > \alpha/2\right\}\right| \\
&\qquad \leq \left|\left\{x \in \mathbb{T}-\Omega^* : \Big(\sum_{k=1}^\infty |Mg_k(x)|^r\Big)^{1/r} > \alpha/6\right\}\right| \\
&\qquad \leq \left|\left\{\Big(\sum_{k=1}^\infty |Mg_k|^r\Big)^{1/r} > \alpha/6\right\}\right| \lesssim \frac{1}{\alpha}\|F\|_1.
\end{aligned}$$

Therefore,

$$\left|\left\{\Big(\sum_{k=1}^{\infty}|Mf_k''|^r\Big)^{1/r}>\alpha/2\right\}\right| = \left|\left\{x\in\mathbb{T}-\Omega^*:\Big(\sum_{k=1}^{\infty}|Mf_k''(x)|^r\Big)^{1/r}>\alpha/2\right\}\right|$$
$$+\left|\left\{x\in\Omega^*:\Big(\sum_{k=1}^{\infty}|Mf_k''(x)|^r\Big)^{1/r}>\alpha/2\right\}\right|$$
$$\lesssim \frac{1}{\alpha}\|F\|_1+|\Omega^*|\lesssim\frac{1}{\alpha}\|F\|_1.$$

Recall $f_k = f_k' + f_k''$, so that $Mf_k \le Mf_k' + Mf_k''$. By Minkowski,

$$\Big(\sum_{k=1}^{\infty}|Mf_k(x)|^r\Big)^{1/r}\le\Big(\sum_{k=1}^{\infty}|Mf_k'(x)|^r\Big)^{1/r}+\Big(\sum_{k=1}^{\infty}|Mf_k''(x)|^r\Big)^{1/r}.$$

Finally, we see

$$\begin{aligned}&\left|\left\{\Big(\sum_{k=1}^{\infty}|Mf_k|^r\Big)^{1/r}>\alpha\right\}\right|\\ &\quad\le\left|\left\{\Big(\sum_{k=1}^{\infty}|Mf_k'|^r\Big)^{1/r}>\alpha/2\right\}\right|+\left|\left\{\Big(\sum_{k=1}^{\infty}|Mf_k''|^r\Big)^{1/r}>\alpha/2\right\}\right|\\ &\quad\lesssim\frac{1}{\alpha}\|F\|_1.\end{aligned}$$

This holds for all $\alpha > \|F\|_1$. But, if $\alpha \le \|F\|_1$, then $|\{(\sum |Mf_k|^r)^{1/r} > \alpha\}| \le 1 \le \|F\|_1/\alpha$ trivially. This completes the proof. $\square$

Theorem 2.12. *For any sequence $f_1, f_2, \ldots$ on $\mathbb{T}$ and $1 < p \le r < \infty$*

$$\left\|\Big(\sum_{k=1}^{\infty}|Mf_k|^r\Big)^{1/r}\right\|_p\lesssim\left\|\Big(\sum_{k=1}^{\infty}|f_k|^r\Big)^{1/r}\right\|_p,$$

where the underlying constants depend only on p and r.

Proof. The case $p = r$ has already been shown in Lemma 2.10. Let $B = \ell^r$, a Banach space. Then, $\mathcal{M}(\mathbb{T}, B)$ is the set of sequences of functions $f = (f_1, f_2, \ldots)$ where each $f_k : \mathbb{T} \to \mathbb{C}$ is measurable. Further, $\|f(x)\|_B = (\sum_k |f_k(x)|^r)^{1/r}$.

Define $\overline{M}$ on $\mathcal{M}(\mathbb{T}, B)$ by $\overline{M}(f_1, f_2, \ldots) = (Mf_1, Mf_2, \ldots)$. Then, $\overline{M}$ is sublinear by Minkowski. Theorem 2.11 says $\overline{M} : L^1_B \to L^{1,\infty}_B$, and Lemma 2.10 says

$\overline{M} : L^r_B \to L^r_B$. It follows then from Theorem 1.11 that $\overline{M} : L^p_B \to L^p_B$ for all $1 < p < r$, which is exactly what we wanted to prove. $\square$

Lemma 2.13. *For any $1 < r < \infty$ and any $f, \phi : \mathbb{T} \to \mathbb{C}$, we have*

$$\int_{\mathbb{T}} |Mf|^r |\phi| \; dx \lesssim \int_{\mathbb{T}} |f|^r M\phi \, dx,$$

where the underlying constants depend only on r.

Proof. Fix $\phi : \mathbb{T} \to \mathbb{C}$. If ϕ is identically 0, there is nothing to prove. So, assume otherwise. We consider the operator M from $(\mathbb{T}, M\phi \; dx)$ to $(\mathbb{T}, |\phi| \; dx)$.

As ϕ is not identically 0, $M\phi > 0$ everywhere. Hence, $\| \cdot \|_\infty = \| \cdot \|_{L^\infty(M\phi \, dx)}$. On the other hand, it is clear that $\| \cdot \|_{L^\infty(|\phi| \, dx)} \leq \| \cdot \|_\infty$. Thus, $\|Mf\|_{L^\infty(|\phi| \, dx)} \leq \|Mf\|_\infty \leq \|f\|_\infty = \|f\|_{L^\infty(M\phi \, dx)}$. Namely, $M : L^\infty(\mathbb{T}, M\phi \; dx) \to L^\infty(\mathbb{T}, |\phi| \; dx)$.

Fix $\alpha > 0$ and $f : \mathbb{T} \to \mathbb{C}$. Consider $\{Mf > \alpha\}$. Assume for the moment that this set is non-empty. By Lemma 2.5, choose disjoint intervals I_j so that $|I_j|^{-1} \int_{I_j} |f| \, dm \geq \alpha/4$ and $\{Mf > \alpha\} \subseteq \bigcup_j I_j^*$. Then,

$$\begin{aligned}
\int_{I_j} f(x) M\phi(x) \, dx &\geq \int_{I_j} f(x) \bigg(\frac{1}{|I_j^*|} \int_{I_j^*} |\phi(y)| \, dy \bigg) \, dx \\
&\geq \frac{1}{3} \bigg(\int_{I_j^*} |\phi(y)| \, dy \bigg) \cdot \bigg(\frac{1}{|I_j|} \int_{I_j} f(x) \, dx \bigg) \\
&\geq \frac{\alpha}{12} \int_{I_j^*} |\phi(y)| \, dy.
\end{aligned}$$

Summing over j, we have

$$\alpha \int_{\{Mf > \alpha\}} |\phi(x)| \, dx \leq 12 \sum_j \int_{I_j} f(x) M\phi(x) \, dx \lesssim \int_{\mathbb{T}} f(x) M\phi(x) \, dx.$$

This holds so long as $\{Mf > \alpha\}$ is non-empty. However, if this set is empty, the above holds trivially. This says $M : L^1(\mathbb{T}, M\phi \; dx) \to L^{1,\infty}(\mathbb{T}, |\phi| \; dx)$.

Therefore, we see $M : L^r(\mathbb{T}, M\phi\ dx) \to L^r(\mathbb{T}, |\phi|\ dx)$ for all $1 < r < \infty$ by the Marcinkiewicz interpolation theorem. This is precisely the statement we wanted to prove. $\square$

Theorem 2.14. *For any sequence $f_1, f_2, \ldots$ on $\mathbb{T}$ and $1 < p, r < \infty$*

$$\Big\|\Big(\sum_{k=1}^{\infty} |Mf_k|^r\Big)^{1/r}\Big\|_p \lesssim \Big\|\Big(\sum_{k=1}^{\infty} |f_k|^r\Big)^{1/r}\Big\|_p,$$

where the underlying constants depend only on p and r.

Proof. The case $1 < p \le r < \infty$ has already been shown in Theorem 2.12.

Fix $1 < r < p < \infty$. Let $q = p/r > 1$ and $\|\phi\|_{q'} \le 1$ (where $1/q + 1/q' = 1$). Then, by Lemma 2.13

$$\begin{aligned}\int_{\mathbb{T}} \sum_{k=1}^{\infty} |Mf_k|^r |\phi|\, dx &\lesssim \int_{\mathbb{T}} \sum_{k=1}^{\infty} |f_k|^r M\phi\, dx \le \Big\|\sum_{k=1}^{\infty} |f_k|^r\Big\|_q \|M\phi\|_{q'} \\ &\lesssim \|\phi\|_{q'} \Big\|\sum_{k=1}^{\infty} |f_k|^r\Big\|_q \le \Big\|\sum_{k=1}^{\infty} |f_k|^r\Big\|_q.\end{aligned}$$

As ϕ in the unit ball of $L^{q'}$ is arbitrary, we have

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^{\infty} |Mf_k|^r\Big)^{1/r}\Big\|_p^r = \Big\|\sum_{k=1}^{\infty} |Mf_k|^r\Big\|_q &= \sup\Big\{\int_{\mathbb{T}} \sum_{k=1}^{\infty} |Mf_k|^r |\phi|\, dx : \|\phi\|_{q'} \le 1\Big\} \\ &\lesssim \Big\|\sum_{k=1}^{\infty} |f_k|^r\Big\|_q = \Big\|\Big(\sum_{k=1}^{\infty} |f_k|^r\Big)^{1/r}\Big\|_p^r\end{aligned}$$

$\square$

2.3 Strong Maximal Operator

There are multiple ways to define maximal operators for functions $f : \mathbb{T}^d \to \mathbb{C}$. If the maximal function is defined to be the supremum over one-parameter "cubes" in

$\mathbb{T}^d$, then it would satisfy all the preceding results by essentially the same arguments. However, we will be most interested in a multi-parameter maximal function. This will require the following definition.

Definition. We say a set $R \subseteq \mathbb{T}^d$ is a rectangle if $R = I_1 \times I_2 \times \ldots \times I_d$, where each I_j is an interval.

Definition. For $f : \mathbb{T}^d \to \mathbb{C}$, define the strong maximal function by

$$M_S f(\vec{x}) = \sup_{\vec{x} \in R} \frac{1}{|R|} \int_R |f(\vec{y})| \, d\vec{y},$$

where the supremum is taken over all rectangles in $\mathbb{T}^d$ containing $\vec{x}$.

It is immediately clear that $\|M_S f\|_\infty \leq \|f\|_\infty$, as before. In addition, M_S satisfies the same $L^p \to L^p$ estimates. To prove this, we take a slight detour.

Denote $\mathcal{M}(\mathbb{T}^d, \mathbb{C})$ the set of measurable functions $f : \mathbb{T}^d \to \mathbb{C}$. For an operator $L : \mathcal{M}(\mathbb{T}, \mathbb{C}) \to \mathcal{M}(\mathbb{T}, \mathbb{C})$, and $1 \leq j \leq d$, define $L_j : \mathcal{M}(\mathbb{T}^d, \mathbb{C}) \to \mathcal{M}(\mathbb{T}^d, \mathbb{C})$ as the operator which applies L to functions with all but the j^{th} variable fixed. Explicitly,

$$L_j f(x_1, \ldots, x_d) = L\big(f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_d)\big)(x_j).$$

Theorem 2.15. *If $L : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ for some $0 < p \leq \infty$, then it follows $L_j : L^p(\mathbb{T}^d) \to L^p(\mathbb{T}^d)$ for all $1 \leq j \leq d$. Similarly, if $L : L^p(\mathbb{T}) \to L^{p,\infty}(\mathbb{T})$ for some $0 < p < \infty$, then $L_j : L^p(\mathbb{T}^d) \to L^{p,\infty}(\mathbb{T}^d)$. Finally, if L satisfies any Fefferman-Stein inequalities on $\mathbb{T}$ for any r and/or p, then L_j satisfies the same inequalities on $\mathbb{T}^d$.*

Proof. For simplicity, we assume $d = 2$ and $j = 1$. Suppose $L : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ with p finite. Let $f : \mathbb{T}^2 \to \mathbb{C}$, and fix $x_2 \in \mathbb{T}$. Write $f_{x_2}(x_1) = f(x_1, x_2)$. Then,

$$\begin{aligned}\int_{\mathbb{T}} |L_1 f(x_1, x_2)|^p \, dx_1 &= \int_{\mathbb{T}} |L(f_{x_2})(x_1)|^p \, dx_1 \\ &\lesssim \int_{\mathbb{T}} |f_{x_2}(x_1)|^p \, dx_1 = \int_{\mathbb{T}} |f(x_1, x_2)|^p \, dx_1.\end{aligned}$$

Integrating in the x_2-variable, we see

$$\|L_1 f\|_{L^p(\mathbb{T}^2)}^p = \int_{\mathbb{T}^2} |L_1 f(x_1, x_2)|^p \, dx_1 \, dx_2 \lesssim \int_{\mathbb{T}^2} |f(x_1, x_2)|^p \, dx_1 \, dx_2 = \|f\|_{L^p(\mathbb{T}^2)}^p.$$

On the other hand, if $p = \infty$, then $|L_1 f(x_1, x_2)| \lesssim \|f(\cdot, x_2)\|_{L^\infty(\mathbb{T})}$ for a.e. x_1. But, $\|f(\cdot, x_2)\|_{L^\infty(\mathbb{T})} \leq \|f\|_{L^\infty(\mathbb{T}^2)}$ for a.e. x_2. Thus, $\|L_1 f\|_{L^\infty(\mathbb{T}^2)} \lesssim \|f\|_{L^\infty(\mathbb{T}^2)}$.

Now suppose $L : L^p(\mathbb{T}) \to L^{p,\infty}(\mathbb{T})$. Then, for any $\lambda > 0$ and any $x_2 \in \mathbb{T}$, we have

$$\lambda^p \big| \big\{ x_1 \in \mathbb{T} : |L_1 f(x_1, x_2)| > \lambda \big\} \big| \lesssim \int_{\mathbb{T}} |f(x_1, x_2)|^p \, dx_1.$$

Integrating

$$\begin{aligned}\lambda^p \big| \big\{ (x_1, x_2) \in \mathbb{T}^2 : |L_1 f(x_1, x_2)| > \lambda \big\} \big| &= \lambda^p \int_{\mathbb{T}} \big| \big\{ x_1 \in \mathbb{T} : |L_1 f(x_1, x_2)| > \lambda \big\} \big| \, dx_2 \\ &\lesssim \int_{\mathbb{T}^2} |f(x_1, x_2)|^p \, dx_1 \, dx_2.\end{aligned}$$

As λ is arbitrary, we have $\|L_1 f\|_{L^{p,\infty}(\mathbb{T}^2)}^p \lesssim \|f\|_{L^p(\mathbb{T}^2)}^p$. Any Fefferman-Stein type inequalities are extended in the same way. $\square$

Applying the definition above to M, consider M_j. Explicitly,

$$M_j f(\vec{x}) = \sup_{x_j \in I} \frac{1}{|I|} \int_I |f(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)| \, dy_j.$$

By the theorem, $M_j : L^p(\mathbb{T}^d) \to L^p(\mathbb{T}^d)$ for all $1 < p \leq \infty$.

On the other hand, fix $\vec{x} \in \mathbb{T}^d$. Let $\epsilon > 0$ and choose a rectangle $\vec{x} \in R$ so that

$$M_S f(\vec{x}) \leq \frac{1}{|R|} \int_R |f(\vec{y})| \, d\vec{y} + \epsilon.$$

Write $R = I_1 \times \ldots \times I_d$, so that $x_j \in I_j$ for each j. Then,

$$\begin{aligned}
M_S f(\vec{x}) - \epsilon &\leq \frac{1}{|I_1| \cdots |I_d|} \int_{I_1 \times \ldots \times I_d} |f(\vec{y})| \, d\vec{y} \\
&= \frac{1}{|I_1|} \int_{I_1} \cdots \frac{1}{|I_d|} \int_{I_d} |f(y_1, \ldots, y_d)| \, dy_d \cdots dy_1 \\
&\leq \frac{1}{|I_1|} \int_{I_1} \cdots \frac{1}{|I_{d-1}|} \int_{I_{d-1}} M_d f(y_1, \ldots, y_{d-1}, x_d) \, dy_{d-1} \cdots dy_1 \\
&\leq M_1 \circ M_2 \circ \cdots \circ M_d f(\vec{x}).
\end{aligned}$$

As ϵ is arbitrary, $M_S f \leq M_1 \circ \cdots \circ M_d f$. From this, it is easily observed that

$$\|M_S f\|_p \leq \|M\|_{L^p \to L^p}^d \|f\|_p$$

for all $1 < p \leq \infty$. However, M_S does not satisfy an $L^1 \to L^{1,\infty}$ estimate. Precisely which set of functions is mapped to weak-L^1 by M_S is the subject of later chapters. For now, we postpone this topic.

Chapter 3
Littlewood-Paley Square Function

In this chapter, we focus on a particular square function of Littlewood-Paley theory [21, 22, 23, 30].

For an adapted family φ_I, define $\phi_I = |I|^{-1/2}\varphi_I$, and note $\|\phi_I\|_2 \lesssim 1$ for all I. Often, ϕ_I is called an L^2-normalized family. Unless otherwise noted, φ_I will always represent an adapted family, and ϕ_I will always represent the L^2-normalization.

For the rest of this chapter, we focus on 0-mean adapted families. For a 0-mean adapted family φ_I and its normalization ϕ_I, define the Littlewood-Paley (discrete) square function by

$$Sf(x) = \left(\sum_I \frac{|\langle \phi_I, f \rangle|^2}{|I|} \chi_I(x) \right)^{1/2},$$

where the sum is over all dyadic intervals. Note that S is sublinear. We are interested in proving $L^p \to L^p$ estimates for this operator. All the underlying norm constants will depend on the original choice of φ_I, and, in particular, the constants C_m. However, for the sake of neatness, we suppress that dependence.

3.1 The L^2 Estimate

Recall the notation $I^n = I + n|I|$. The "canonical" representation is I^n where $|n| \leq 1/2|I|$. That is, the smallest $|n|$ giving this set.

Lemma 3.1. *For any 0-mean adapted family and any integer $|n| \leq 1/2|I|$, $|\langle \phi_I, \phi_{I^n} \rangle| \lesssim \frac{1}{(|n|+1)^2}$.*

Proof. First, if $|n| \leq 1$, then $|\langle \phi_I, \phi_{I^n} \rangle| \leq \|\phi_I\|_2 \|\phi_{I^n}\|_2 \lesssim 1 \leq 4\frac{1}{(|n|+1)^2}$. So, assume $|n| > 1$.

Suppose, for simplicity, that $n > 0$. The other case follows in the same manner. If $|I| = 2^{-k}$, set $N = 2^{k-1}$, so that $\mathbb{T} = \bigcup\{I^m : -N+1 \leq m \leq N\}$, and this union is disjoint. Set $\alpha(n) = \frac{n-1}{2}$ if n is odd and $\frac{n}{2}$ if n is even, so that $\alpha(n)$ is a positive integer, which is strictly less that n. Observe,

$$\begin{aligned}
|\langle \phi_I, \phi_{I^n}\rangle| &= \frac{1}{|I|}\left|\int_{\mathbb{T}} \varphi_I(x)\overline{\varphi_{I^n}(x)}\,dx\right| = \frac{1}{|I|}\left|\sum_{m=-N+1}^{N}\int_{I^m} \varphi_I(x)\overline{\varphi_{I^n}(x)}\,dx\right| \\
&\leq \frac{1}{|I|}\sum_{m=-N+1}^{N}\int_{I^m} |\varphi_I(x)||\varphi_{I^n}(x)|\,dx \\
&\lesssim \frac{1}{|I|}\sum_{m=-N+1}^{N}\int_{I^m} \left(1+\frac{\operatorname{dist}(x,I)}{|I|}\right)^{-3}\left(1+\frac{\operatorname{dist}(x,I^n)}{|I|}\right)^{-3} dx \\
&\leq \sum_{m=-N+1}^{N}\left(1+\frac{\operatorname{dist}(I^m,I)}{|I|}\right)^{-3}\left(1+\frac{\operatorname{dist}(I^m,I^n)}{|I|}\right)^{-3}.
\end{aligned}$$

It is clear that

$$\frac{\operatorname{dist}(I,I^m)}{|I|} = |m|-1 \ (m \neq 0), \qquad \frac{\operatorname{dist}(I^n,I^m)}{|I|} = \min\{|n-m|,|n+m|\}-1 \ (m \neq n).$$

Therefore,

$$\begin{aligned}
|\langle \phi_I, \phi_{I^n}\rangle| &\lesssim \sum_{m=-N+1}^{N}\left(1+\frac{\operatorname{dist}(I^m,I)}{|I|}\right)^{-3}\left(1+\frac{\operatorname{dist}(I^m,I^n)}{|I|}\right)^{-3} \\
&\leq \sum_{|m|\leq\alpha(n)}\left(1+\frac{\operatorname{dist}(I^m,I^n)}{|I|}\right)^{-3} + \sum_{\alpha(n)<|m|\leq N}\left(1+\frac{\operatorname{dist}(I^m,I)}{|I|}\right)^{-3} \\
&= \sum_{|m|\leq\alpha(n)}\frac{1}{\min(|n+m|,|n-m|)^3} + \sum_{\alpha(n)<|m|\leq N}\frac{1}{|m|^3} \\
&\leq 2\sum_{m=0}^{\alpha(n)}\frac{1}{|n-m|^3} + 2\sum_{m=\alpha(n)}^{N}\frac{1}{m^3} \leq 2\sum_{m=\alpha(n)}^{n}\frac{1}{m^3} + 2\sum_{m=\alpha(n)}^{N}\frac{1}{m^3} \\
&\leq 4\sum_{m=\alpha(n)}^{\infty}\frac{1}{m^3} \lesssim \frac{1}{\alpha(n)^2} \lesssim \frac{1}{(n+1)^2}.
\end{aligned}$$

$\square$

Let I be a dyadic interval with $|I| = 2^{-k}$. Then, for $1 \leq j \leq k-1$, let J be the unique dyadic interval containing I with $|J| = 2^j|I|$. For $|n| \leq 1/(2|J|)$, denote $I(j,n) = J^n$. That is, for an interval I, $I(j,n)$ is the interval obtained by enlarging to the dyadic interval of length $2^j|I|$ and shifting n units of the new length.

Lemma 3.2. *For any 0-mean adapted family with j and n as above, $|\langle \phi_I, \phi_{I(j,n)} \rangle| \lesssim 2^{-j}\frac{1}{(|n|+1)^2}$.*

Proof. Suppose $|I| = 2^{-k}$. Let J be the dyadic interval containing I with $|J| = 2^j|I|$. Then, $J^n = I(j,n)$. Set $N = 2^{k-j-1}$ so that $\mathbb{T} = \bigcup\{J^m : -N+1 \leq m \leq N\}$ and $\mathbb{T} = \bigcup\{I^m : -2^jN+1 \leq m \leq 2^jN\}$, and these unions are disjoint.

For a moment, let us think of φ_I as a periodic function on the real line. Let I' be an interval in $\mathbb{T}$, which can be thought of as an interval on the real line contained in $[0,1]$. Then, for $x, z \in I'$, we have by the mean value theorem that $|\varphi_I(x) - \varphi_I(z)| = |\varphi_I'(z_x)||x-z| \leq |\varphi_I'(z_x)||I'|$, for some z_x in I'. Thus, if we fix a z^m in each I^m, as φ_I has integral 0,

$$
\begin{aligned}
|\langle \phi_I, \phi_{I(j,n)} \rangle| &\leq \frac{1}{|I|^{1/2}} \frac{1}{|J|^{1/2}} \sum_{m=-2^jN+1}^{2^jN} \left| \int_{I^m} \varphi_I(x) \overline{\varphi_{J^n}(x)}\, dx \right| \\
&= 2^{-j/2} \frac{1}{|I|} \sum_{m=-2^jN+1}^{2^jN} \left| \int_{I^m} \varphi_I(x) \left[\overline{\varphi_{J^n}(x)} - \overline{\varphi_{J^n}(z^m)} \right] dx \right| \\
&\leq 2^{-j/2} \sum_{m=-2^jN+1}^{2^jN} \int_{I^m} |\varphi_I(x)||\varphi_{J^n}'(z_x^m)|\, dx \\
&\lesssim 2^{-j/2} \sum_{m=-2^jN+1}^{2^jN} \frac{|I^m|}{|J^n|} \left(1 + \frac{\operatorname{dist}(I^m, I)}{|I|}\right)^{-4} \left(1 + \frac{\operatorname{dist}(I^m, J^n)}{|J|}\right)^{-10} \\
&= 2^{-3j/2} \sum_{m=-2^jN+1}^{2^jN} \left(1 + \frac{\operatorname{dist}(I^m, I)}{|I|}\right)^{-4} \left(1 + \frac{\operatorname{dist}(I^m, J^n)}{|J|}\right)^{-10}
\end{aligned}
$$

Hence, if $|n| \leq 1$, then

$$|\langle \phi_I, \phi_{I(j,n)} \rangle| \lesssim 2^{-3j/2} \sum_{m=-2^jN+1}^{2^jN} \left(1 + \frac{\operatorname{dist}(I^m, I)}{|I|}\right)^{-4}$$
$$\leq 2^{-3j/2}\left[1 + 2\sum_{m=1}^{2^jN} \frac{1}{m^4}\right] \leq 2^{-3j/2}\left[1 + 2\sum_{m=1}^{\infty} \frac{1}{m^4}\right]$$
$$\lesssim 2^{-3j/2} \leq 2^{-j} \leq 4 \cdot 2^{-j} \frac{1}{(|n|+1)^2}.$$

Therefore, assume $|n| > 1$. As before, consider only the $n > 0$ case, as the other is done in the same way. Let $\alpha(n)$ be as previously defined. First, we see

$$\sum_{2^j\alpha(n) < |m| \leq 2^jN} \left(1 + \frac{\operatorname{dist}(I^m, I)}{|I|}\right)^{-4} \left(1 + \frac{\operatorname{dist}(I^m, J^n)}{|J|}\right)^{-10} \leq$$
$$\sum_{2^j\alpha(n) < |m| \leq 2^jN} \left(1 + \frac{\operatorname{dist}(I^m, I)}{|I|}\right)^{-4} \leq 2 \sum_{m=2^j\alpha(n)}^{2^jN} \frac{1}{m^4} \leq$$
$$2 \sum_{m=\alpha(n)}^{\infty} \frac{1}{m^4} \lesssim \frac{1}{\alpha(n)^2} \lesssim \frac{1}{(|n|+1)^2}.$$

On the other hand, by Hölder, we have

$$\sum_{|m| \leq 2^j\alpha(n)} \left(1 + \frac{\operatorname{dist}(I^m, I)}{|I|}\right)^{-4} \left(1 + \frac{\operatorname{dist}(I^m, J^n)}{|J|}\right)^{-10} \leq$$
$$\left(\sum_{|m| \leq 2^j\alpha(n)} \left(1 + \frac{\operatorname{dist}(I^m, I)}{|I|}\right)^{-2}\right)^{1/2} \cdot \left(\sum_{|m| \leq 2^j\alpha(n)} \left(1 + \frac{\operatorname{dist}(I^m, J^n)}{|J|}\right)^{-5}\right)^{1/2} \leq$$
$$\left(1 + 2\sum_{m=1}^{\infty} \frac{1}{m^2}\right)^{1/2} \cdot \left(\sum_{|m| \leq 2^j\alpha(n)} \left(1 + \frac{\operatorname{dist}(I^m, J^n)}{|J|}\right)^{-5}\right)^{1/2} \lesssim$$
$$\left(\sum_{|m| \leq 2^j\alpha(n)} \left(1 + \frac{\operatorname{dist}(I^m, J^n)}{|J|}\right)^{-5}\right)^{1/2}.$$

For each $|m| \leq 2^j\alpha(n)$, there is an m' so that $I^m \subset J^{m'}$ and $|m'| \leq \alpha(n)$. Further, there are exactly 2^j of these I^m contained in each $J^{m'}$. Thus,

$$\left(\sum_{|m|\le 2^j\alpha(n)}\left(1+\frac{\mathrm{dist}(I^m,J^n)}{|J|}\right)^{-5}\right)^{1/2}\le\left(2^j\sum_{|m|\le\alpha(n)}\left(1+\frac{\mathrm{dist}(J^m,J^n)}{|J|}\right)^{-5}\right)^{1/2}=$$

$$\left(2^j\sum_{|m|\le\alpha(n)}\frac{1}{\min(|n+m|,|n-m|)^5}\right)^{1/2}\lesssim\left(2^j\sum_{m=0}^{\alpha(n)}\frac{1}{|n-m|^5}\right)^{1/2}\le$$

$$\left(2^j\sum_{m=\alpha(n)}^{\infty}\frac{1}{m^5}\right)^{1/2}\lesssim 2^{j/2}\frac{1}{\alpha(n)^2}\lesssim 2^{j/2}\frac{1}{(|n|+1)^2}.$$

Finally, combining all of this, we have

$$|\langle\phi_I,\phi_{I(j,n)}\rangle|\lesssim 2^{-3j/2}\sum_{m=-2^jN+1}^{2^jN}\left(1+\frac{\mathrm{dist}(I^m,I)}{|I|}\right)^{-4}\left(1+\frac{\mathrm{dist}(I^m,J^n)}{|J|}\right)^{-10}$$

$$\lesssim 2^{-3j/2}\Big[\frac{1}{(|n|+1)^2}+2^{j/2}\frac{1}{(|n|+1)^2}\Big]\lesssim 2^{-j}\frac{1}{(|n|^2+1)}.$$

□

For any $N\in\mathbb{N}$, define the linear operator L_N by

$$L_Nf(x)=\sum_{|I|\ge 2^{-N}}\langle\phi_I,f\rangle\overline{\phi_I(x)}.$$

The following is the crucial estimate in our desired L^2 result.

Lemma 3.3. *For any 0-mean adapted family and any function $f:\mathbb{T}\to\mathbb{C}$,*

$$\|L_Nf\|_2^2\lesssim\sum_{|I|\ge 2^{-N}}|\langle\phi_I,f\rangle|^2,$$

where the underlying constant is independent of N and f.

Proof. We note that

$$
\begin{aligned}
\|L_N f\|_2^2 &= \int_{\mathbb{T}} L_N f(x)\overline{L_N f(x)}\,dx \\
&= \int_{\mathbb{T}} \Big[\sum_{|I|\geq 2^{-N}} \langle \phi_I, f\rangle \overline{\phi_I(x)}\Big]\Big[\sum_{|J|\geq 2^{-N}} \overline{\langle \phi_J, f\rangle}\phi_J(x)\Big]\,dx \\
&= \sum_{|I|,|J|\geq 2^{-N}} \langle \phi_I, f\rangle \overline{\langle \phi_J, f\rangle}\langle \phi_J, \phi_I\rangle \\
&\leq \sum_{|I|,|J|\geq 2^{-N}} |\langle \phi_I, f\rangle||\langle \phi_J, f\rangle||\langle \phi_I, \phi_J\rangle|.
\end{aligned}
$$

We break this sum into three pieces: the terms where $|I| = |J|$, where $|I| < |J|$, and where $|J| < |I|$. The last two pieces are symmetric, and we only prove one of them. For the first piece,

$$
\sum_{|I|=|J|\geq 2^{-N}} |\langle \phi_I, f\rangle||\langle \phi_J, f\rangle||\langle \phi_J, \phi_I\rangle|
$$
$$
= \sum_{|I|\geq 2^{-N}} \sum_{n=-1/(2|I|)+1}^{1/(2|I|)} |\langle \phi_I, f\rangle||\langle \phi_{I^n}, f\rangle||\langle \phi_I, \phi_{I^n}\rangle|.
$$

For the purposes of this proof only, we adopt a notational convention. For an interval I and integer $|n| \leq 1/(2|I|)$, let I^n be as normal, and ϕ_{I^n} the adapted family member for this interval. But, for n not satisfying this property, let ϕ_{I^n} be identically 0. Then, by Lemma 3.1, $|\langle \phi_I, \phi_{I^n}\rangle| \lesssim (|n|+1)^{-2}$ for all n. Further, we can write

$$
\begin{aligned}
\sum_{|I|=|J|\geq 2^{-N}} |\langle \phi_I, f\rangle||\langle \phi_J, f\rangle||\langle \phi_I, \phi_J\rangle| &= \sum_{n\in\mathbb{Z}} \sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle||\langle \phi_{I^n}, f\rangle||\langle \phi_I, \phi_{I^n}\rangle| \\
&\lesssim \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^2} \sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle||\langle \phi_{I^n}, f\rangle| \\
&\leq \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^2} \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg)^{1/2} \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_{I^n}, f\rangle|^2 \Bigg)^{1/2} \\
&= \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg) \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^2} \\
&\lesssim \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg).
\end{aligned}
$$

The transition from the fourth to fifth line follows because for a fixed n, summing over all I^n is equivalent to summing over all I. The shift is irrelevant in this regard.

Now let us focus on the case $|I| < |J|$. Again, we adopt here some unusual notation. For appropriate j and n, let $I(j,n)$ be as defined before and $\phi_{I(j,n)}$ as normal. If either j or n is not small enough with respect to I, then set $\phi_{I(j,n)}$ to be 0. Then, by Lemma 3.2, $|\langle \phi_I, \phi_{I(j,n)}\rangle| \lesssim 2^{-j}(|n|+1)^{-2}$ for all j and n. Further,

$$
\begin{aligned}
&\sum_{|J|>|I|\geq 2^{-N}} |\langle \phi_I, f\rangle||\langle \phi_J, f\rangle||\langle \phi_I, \phi_J\rangle| \\
&\qquad = \sum_{k=1}^{N} \sum_{|I|=2^{-k}} \sum_{j=1}^{k-1} \sum_{n=-2^{k-j-1}+1}^{2^{k-j-1}} |\langle \phi_I, f\rangle||\langle \phi_{I(j,n)}, f\rangle||\langle \phi_I, \phi_{I(j,n)}\rangle| \\
&\qquad = \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{N}} \sum_{k=1}^{N} \sum_{|I|=2^{-k}} |\langle \phi_I, f\rangle||\langle \phi_{I(j,n)}, f\rangle||\langle \phi_I, \phi_{I(j,n)}\rangle| \\
&\qquad \lesssim \sum_{j\in\mathbb{N}} 2^{-j} \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^2} \sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle||\langle \phi_{I(j,n)}, f\rangle| \\
&\qquad \leq \sum_{j\in\mathbb{N}} 2^{-j} \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^2} \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg)^{1/2} \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_{I(j,n)}, f\rangle|^2 \Bigg)^{1/2}.
\end{aligned}
$$

Fix j and n, and consider a dyadic interval $|J| \geq 2^{-N}$. One of two things is true. Either there are no $|I| \geq 2^{-N}$ such that $J = I(j,n)$, due to the incompatibility of

j, n, and/or N. Or, there are exactly 2^j such I. Indeed, if there is an I such that $I \subset J_0$, where $|J_0| = |J|$ and $J_0^n = J$, then $J = I(j, n)$ for all I contained in this J_0. Hence,

$$\begin{aligned}
&\sum_{|J|>|I|\geq 2^{-N}} |\langle \phi_I, f\rangle||\langle \phi_J, f\rangle||\langle \phi_I, \phi_J\rangle| \\
&\qquad \lesssim \sum_{j\in\mathbb{N}} 2^{-j} \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^2} \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg)^{1/2} \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_{I(j,n)}, f\rangle|^2 \Bigg)^{1/2} \\
&\qquad \leq \sum_{j\in\mathbb{N}} 2^{-j} \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^2} \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg)^{1/2} \Bigg(2^j \sum_{|J|\geq 2^{-N}} |\langle \phi_J, f\rangle|^2 \Bigg)^{1/2} \\
&\qquad = \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg) \sum_{j\in\mathbb{N}} 2^{-j/2} \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^2} \\
&\qquad \lesssim \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg).
\end{aligned}$$

□

Theorem 3.4. *For any 0-mean adapted family, $S : L^2 \to L^2$.*

Proof. Let $f \in L^2$ and fix $N \in \mathbb{N}$. First, we note

$$\begin{aligned}
\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 &\leq \sum_{|I|\geq 2^{-N}} \|f\|_2^2 \|\phi_I\|_2^2 \lesssim \|f\|_2^2 \sum_{|I|\geq 2^{-N}} 1 \\
&= \|f\|_2^2 \Big(2 + 2^2 + \ldots + 2^N \Big) \leq 2^{N+1} \|f\|_2^2 < \infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 &= \sum_{|I|\geq 2^{-N}} \langle \phi_I, f\rangle \overline{\langle \phi_I, f\rangle} = \Big\langle \sum_{|I|\geq 2^{-N}} \langle \phi_I, f\rangle \overline{\phi_I}, \overline{f} \Big\rangle \\
&= \langle L_N f, \overline{f}\rangle \leq \|L_N f\|_2 \|f\|_2 \lesssim \|f\|_2 \Bigg(\sum_{|I|\geq 2^{-N}} |\langle \phi_I, f\rangle|^2 \Bigg)^{1/2}
\end{aligned}$$

implies

$$\left(\sum_{|I| \geq 2^{-N}} |\langle \phi_I, f \rangle|^2 \right)^{1/2} \lesssim \|f\|_2.$$

As N is arbitrary, and the bounds do not depend on N, let N tend to infinity. Then,

$$\|Sf\|_2^2 = \int_{\mathbb{T}} \sum_I \frac{|\langle \phi_I, f \rangle|^2}{|I|} \chi_I(x)\, dx = \sum_I |\langle \phi_I, f \rangle|^2 \lesssim \|f\|_2^2.$$

$\square$

3.2 The Weak-L^1 Estimate

Lemma 3.5. *Let $f \in L^1(\mathbb{T})$ and $\alpha > \|f\|_1$ a constant. Then, there exists a sequence of disjoint dyadic intervals $I_1, I_2, \ldots$, with $\Omega = \bigcup_k I_k$, and a decomposition $f = g + b$, $b = \sum_k b_k$, such that*

$$\|g\|_2^2 \lesssim \alpha \|f\|_1,$$
$$\operatorname{supp}(b_k) \subseteq I_k, \;\; \|b_k\|_1 \lesssim \alpha |I_k|, \;\; \int_{\mathbb{T}} b_k(x)\, dx = 0,$$
$$|\Omega| = \sum_{k=1}^{\infty} |I_k| \leq \frac{\|f\|_1}{\alpha}.$$

Proof. Define $\Omega = \{M_D f > \alpha\}$. As $|f| \leq M_D f$ a.e., we see immediately that $|f| \leq M_D f \leq \alpha$ a.e. on Ω^c. If Ω is empty, then $|f| \leq \alpha$ a.e. on $\mathbb{T}$. Simply set $g = f$, $b_k = 0$, and I_k empty for each k. Then, the conditions are trivially satisfied.

Now, assume Ω is not empty. Let $\mathcal{D}$ be the countable collection of all dyadic intervals I such that $\frac{1}{|I|} \int_I |f(y)|\, dy > \alpha$. By construction, $\Omega = \bigcup_{\mathcal{D}} I$. We say a dyadic interval $I \in \mathcal{D}$ is maximal if for every $I' \in \mathcal{D}$, we have either $I' \subseteq I$ or I, I' are disjoint. Clearly, every $I \in \mathcal{D}$ is contained in a maximal interval. Let $I_1, I_2, \ldots$

be the maximal intervals of $\mathcal{D}$, which are necessarily disjoint. Further, it is clear that

$$\Omega = \bigcup_{\mathcal{D}} I = \bigcup_{\mathbb{N}} I_k.$$

As each $I_k \in \mathcal{D}$, we have $\alpha|I_k| < \int_{I_k} |f(y)|\,dy$. As the I_k are disjoint, simply sum over k to see $\alpha|\Omega| \leq \int_{\Omega} |f(y)|\,dy \leq \|f\|_1$. On the other hand, if $|I_k| < 1/2$, then there is some dyadic interval I_k' which contains I_k and satisfies $|I_k'| = 2|I_k|$. But, $I_k' \notin \mathcal{D}$, because otherwise I_k could not be maximal. Thus, $\alpha|I_k'| \geq \int_{I_k'} |f(y)|\,dy$, which implies $\int_{I_k} |f(y)|\,dy \leq \int_{I_k'} |f(y)|\,dy \leq \alpha|I_k'| = 2\alpha|I_k|$. Similarly, if $|I_k| = 1/2$, then $\int_{I_k} |f(y)|\,dy \leq \|f\|_1 < \alpha = 2\alpha|I_k|$.

Define the function g by

$$g(x) = f(x)\chi_{\Omega^c}(x) + \sum_k \Big(\frac{1}{|I_k|}\int_{I_k} f(y)\,dy\Big)\chi_{I_k}(x).$$

It is easily seen that $g(x)^2 = f(x)^2\chi_{\Omega^c}(x) + \sum_k (\frac{1}{|I_k|}\int_{I_k} f(y)\,dy)^2 \chi_{I_k}(x)$. Thus,

$$\begin{aligned}
\|g\|_2^2 &= \int_{\Omega^c} |f(x)|^2\,dx + \sum_k \frac{1}{|I_k|}\Big(\int_{I_k} f(y)\,dy\Big)^2 \\
&\leq \int_{\Omega^c} \alpha|f(x)|\,dx + \sum_k 4\alpha^2|I_k| \\
&\leq \alpha\|f\|_1 + 4\alpha^2|\Omega| \leq 5\alpha\|f\|_1.
\end{aligned}$$

Set $b = f - g$ and $b_k = (f - \frac{1}{|I_k|}\int_{I_k} f(y)\,dy)\chi_{I_k}$. Then, we immediately have $\int b_k(x)\,dx = 0$. Further, each b_k is supported on I_k and $b = \sum_k b_k$. Finally,

$$\|b_k\|_1 = \int_{I_k} \left| f(x) - \frac{1}{|I_k|}\int_{I_k} f(y)\,dy \right| dx \leq 2\int_{I_k} |f(x)|\,dx \leq 4\alpha|I_k|.$$

□

Lemma 3.6. *If $a : \mathbb{T} \to \mathbb{C}$ is in L^1, supported in some dyadic interval I, and satisfies $\int_{\mathbb{R}} a(x)\,dx = 0$, then $\|Sa\|_{L^1(\mathbb{T}-2I)} \lesssim \|a\|_1$.*

Proof. If $|I| = 1/2$, then $2I = \mathbb{T}$, and the result is trivially satisfied. So, assume $|I| < 1/2$. Pick a dyadic interval J such that $|J| < |I|$. Note, either $J \subset 2I$ or J and $2I$ are disjoint. Assume it is the later, i.e. $J \not\subseteq 2I$. Then,

$$\begin{aligned}
\frac{|\langle \phi_J, a\rangle|}{|J|^{1/2}} &\le \frac{1}{|J|^{1/2}} \int_I |a(x)||\phi_J(x)|\,dx = \frac{1}{|J|}\int_I |a(x)||\varphi_J(x)|\,dx \\
&\lesssim \frac{1}{|J|}\int_I |a(x)|\left(1 + \frac{\operatorname{dist}(x,J)}{|J|}\right)^{-2} dx \\
&\le \frac{1}{|J|}\|a\|_1\left(1 + \frac{\operatorname{dist}(I,J)}{|J|}\right)^{-2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\| \sum_{|J|<|I|} \frac{|\langle \phi_J, a\rangle|}{|J|^{1/2}} \chi_J \right\|_{L^1(\mathbb{T}-2I)} &\lesssim \|a\|_1 \sum_{|J|<|I|,\, J\not\subseteq 2I} \left(1 + \frac{\operatorname{dist}(J,I)}{|J|}\right)^{-2} \\
&= \|a\|_1 \sum_{j=1}^{\infty}\left[\sum_{J\not\subseteq 2I,\, |J|=2^{-j}|I|}\left(1 + \frac{\operatorname{dist}(J,I)}{|J|}\right)^{-2}\right].
\end{aligned}$$

Consider the dyadic intervals J so that $|J| = 2^{-j}|I|$ and $J \not\subseteq 2I$. The minimum value of $\operatorname{dist}(J,I)$ for all such J is $|I|/2$, due to the definition of $2I$. But, $|I|/2 = 2^{j-1}|J|$. There are two such J, on either side of $2I$. Taking one step further from $2I$, there are two J with $\operatorname{dist}(J,I) = (2^{j-1}+1)|J|$. Taking another step, there are two J with $\operatorname{dist}(J,I) = (2^{j-1}+2)|J|$, and so on, until we have exhausted $\mathbb{T}$. Thus,

$$\sum_{J\not\subseteq 2I,\, |J|=2^{-j}|I|}\left(1 + \frac{\operatorname{dist}(J,I)}{|J|}\right)^{-2} \le 2\sum_{i=2^{j-1}}^{\infty}(1+i)^{-2} \le 2^{2-j},$$

and

$$\left\| \sum_{|J|<|I|} \frac{|\langle \phi_J, a\rangle|}{|J|^{1/2}} \chi_J \right\|_{L^1(\mathbb{T}-2I)} \lesssim \|a\|_1 \sum_{j=1}^{\infty} 2^{-j} = \|a\|_1.$$

Now, let J be a dyadic interval with $|J| \geq |I|$. Fix $z \in I$. As in the proof of Lemma 3.2, by the mean value theorem, for all $x \in I$ there exists a $z_x \in I$ such that $|\varphi_J(x) - \varphi_J(z)| \leq |\varphi_J'(z_x)||I|$. Recalling that the integral of a is 0, we have

$$\begin{aligned}\frac{|\langle \phi_J, a\rangle|}{|J|^{1/2}} &= \frac{1}{|J|}\Big|\int_I \varphi_J(x)\overline{a(x)}\,dx\Big| = \frac{1}{|J|}\Big|\int_I \overline{a(x)}\big(\varphi_J(x) - \varphi_J(z)\big)\,dx\Big| \\ &\leq \frac{|I|}{|J|}\int_I |a(x)||\varphi_J'(z_x)|\,dx \lesssim \|a\|_1 \frac{|I|}{|J|^2}\left(1 + \frac{\operatorname{dist}(J,I)}{|J|}\right)^{-2}.\end{aligned}$$

So,

$$\begin{aligned}\Big\| \sum_{|J|\geq|I|} \frac{|\langle \phi_J, a\rangle|}{|J|^{1/2}}\chi_J \Big\|_{L^1(\mathbb{T}-2I)} &\leq \Big\| \sum_{|J|\geq|I|} \frac{|\langle \phi_J, a\rangle|}{|J|^{1/2}}\chi_J \Big\|_1 \\ &\lesssim \|a\|_1 \sum_{|J|\geq|I|} \frac{|I|}{|J|}\left(1 + \frac{\operatorname{dist}(J,I)}{|J|}\right)^{-2} \\ &= \|a\|_1 \sum_{j=0}^{k-1}\left[\sum_{|J|=2^j|I|} \frac{1}{2^j}\left(1 + \frac{\operatorname{dist}(J,I)}{|J|}\right)^{-2}\right],\end{aligned}$$

if $|I| = 2^{-k}$. Consider the J with $|J| = 2^j|I|$. There is one such interval J' with $I \subseteq J'$. Now, for every other such J, we have $\operatorname{dist}(J,I) \geq \operatorname{dist}(J,J')$. There are three such J (including J') with $\operatorname{dist}(J,J') = 0$. Moving farther to the left and right, there are two with $\operatorname{dist}(J,J') = |J|$, two with $\operatorname{dist}(J,J') = 2|J|$, and so on, until we exhaust all such J. Thus,

$$\sum_{|J|=2^j|I|}\left(1 + \frac{\operatorname{dist}(J,I)}{|J|}\right)^{-2} \leq \sum_{|J|=2^j|I|}\left(1 + \frac{\operatorname{dist}(J,J')}{|J|}\right)^{-2} \leq 3 + 2\sum_{i=1}^{\infty}(1+i)^{-2} \leq 5,$$

and

$$\Big\| \sum_{|J|\geq|I|} \frac{|\langle \phi_J, a\rangle|}{|J|^{1/2}}\chi_J \Big\|_{L^1(\mathbb{T}-2I)} \lesssim \|a\|_1 \sum_{j=0}^{k-1} 2^{-j} \leq \|a\|_1 \sum_{j=0}^{\infty} 2^{-j} = 2\|a\|_1.$$

Recalling that the ℓ^2-norm is always less than or equal to the ℓ^1-norm, it follows

$$\|Sa\|_{L^1(\mathbb{T}-2I)} \leq \left\| \sum_J \frac{|\langle a, \phi_J \rangle|}{|J|^{1/2}} \chi_J \right\|_{L^1(\mathbb{T}-2I)} \lesssim \|a\|_1.$$

□

Theorem 3.7. *For any 0-mean adapted family, $S : L^1 \to L^{1,\infty}$.*

Proof. Let $f \in L^1(\mathbb{T})$ and $\alpha \leq \|f\|_1$. Then, $|\{Sf > \alpha\}| \leq 1 \leq \|f\|_1/\alpha$. Now take $\alpha > \|f\|_1$. Apply Lemma 3.5 to find disjoint dyadic intervals I_k and write $f = g + b$. Then, by Chebyshev

$$|\{Sg > \alpha/2\}| \lesssim \frac{1}{\alpha^2}\|Sg\|_2^2 \lesssim \frac{1}{\alpha^2}\|g\|_2^2 \lesssim \frac{1}{\alpha}\|f\|_1.$$

Applying Lemma 3.6 to each b_k, we see $\|Sb_k\|_{L^1(\mathbb{T}-2I_k)} \lesssim \|b_k\|_1 \lesssim \alpha|I_k|$. Define $\Omega^* = \bigcup_k 2I_k$, and note $|\Omega^*| \leq \sum_k |2I_k| = 2\sum_k |I_k| \lesssim \|f\|_1/\alpha$. As S is sublinear,

$$\begin{aligned} |\{Sb > \alpha/2\}| &\leq |\{x \in \Omega^* : Sb(x) > \alpha/2\} + |\{x \in \mathbb{T} - \Omega^* : Sb(x) > \alpha/2\}| \\ &\leq |\Omega^*| + \frac{2}{\alpha}\|Sb\|_{L^1(\mathbb{T}-\Omega^*)} \lesssim \frac{1}{\alpha}\|f\|_1 + \frac{2}{\alpha}\sum_k \|Sb_k\|_{L^1(\mathbb{T}-2I_k)} \\ &\leq \frac{1}{\alpha}\|f\|_1 + 2\sum_k |I_k| \lesssim \frac{\|f\|_1}{\alpha}. \end{aligned}$$

Hence,

$$|\{Sf > \alpha\}| \leq |\{Sg > \alpha/2\}| + |\{Sb > \alpha/2\}| \lesssim \frac{\|f\|_1}{\alpha}.$$

As α is arbitrary, this completes the proof. □

3.3 The Linearization T_ϵ

In order to complete the L^p estimates of S, it is necessary to consider a kind of linearization. Let φ_I^1, φ_I^2 be two 0-mean adapted families. Let ϵ_I be a sequence of scalars, indexed by the dyadic intervals, which is uniformly bounded. Define the linear operator T_ϵ by

$$T_\epsilon f(x) = \sum_I \epsilon_I \langle \phi_I^1, f\rangle \phi_I^2(x),$$

where ϕ_I^1, ϕ_I^2 are, of course, the corresponding normalized families. By dividing out a constant, we can assume $|\epsilon_I| \leq 1$. Our first goal will be to prove T_ϵ maps L^2 to L^2. This follows easily using what we know about S.

Theorem 3.8. *For any 0-mean adapted families, $T_\epsilon : L^2 \to L^2$, where the underlying constant is independent of the sequence ϵ.*

Proof. Fix a sequence (ϵ_I) where $|\epsilon_I| \leq 1$ for all I. Let φ_I^1, φ_I^2 be two 0-mean adapted families, and S^1, S^2 the associated square functions. Fix $f \in L^2(\mathbb{T})$. Let $\|g\|_2 \leq 1$. Then, by two applications of Hölder,

$$\begin{aligned}
|\langle T_\epsilon f, g\rangle| &= \Big|\sum_I \epsilon_I \langle \phi_I^1, f\rangle\langle \phi_I^2, g\rangle\Big| \leq \sum_I |\langle \phi_I^1, f\rangle||\langle \phi_I^2, g\rangle| \\
&= \int_{\mathbb{T}} \sum_I \frac{|\langle \phi_I^1, f\rangle|}{|I|^{1/2}} \frac{|\langle \phi_I^2, g\rangle|}{|I|^{1/2}} \chi_I(x)\, dx \\
&\leq \int_{\mathbb{T}} \Big(\sum_I \frac{|\langle \phi_I^1, f\rangle|^2}{|I|}\chi_I(x)\Big)^{1/2} \Big(\sum_I \frac{|\langle \phi_I^2, g\rangle|^2}{|I|}\chi_I(x)\Big)^{1/2} dx \\
&= \int_{\mathbb{T}} S^1 f(x) S^2 g(x)\, dx \leq \|S^1 f\|_2 \|S^2 g\|_2 \lesssim \|f\|_2\|g\|_2 \leq \|f\|_2.
\end{aligned}$$

As g in the unit ball of L^2 is arbitrary, we see

$$\|T_\epsilon f\|_2 = \sup\Big\{|\langle T_\epsilon f, g\rangle| : \|g\|_2 \leq 1\Big\} \lesssim \|f\|_2.$$

□

Next, we will show T_ϵ maps L^1 into weak-L^1. First, we prove a useful "dualizaton" of weak-L^p.

Lemma 3.9. *Fix $0 < p < \infty$ and $f : \mathbb{T} \to \mathbb{C}$. Suppose that for every set $|E| > 0$ in $\mathbb{T}$, we can choose a subset $E' \subseteq E$ with $|E'| > |E|/2$ and $|\langle f, \chi_{E'} \rangle| \le A|E|^{1-1/p}$. Then, $\|f\|_{p,\infty} \lesssim A$. Conversely, if $\|f\|_{p,\infty} \le A$, then for any set $|E| > 0$ there exists $E' \subseteq E$ with $|E'| > |E|/2$ and $|\langle f, \chi_{E'} \rangle| \lesssim A|E|^{1-1/p}$.*

Proof. Start with the first statement. Fix $\lambda > 0$. Let $E_1 = \{\operatorname{Re} f > \lambda\}$. If $|E_1| = 0$, then clearly $\lambda^p |E_1| \le (2A)^p$. Otherwise, choose $E' \subset E_1$ as per the hypothesis. Now, $|\langle \operatorname{Re} f, \chi_{E'} \rangle| = |\int_{E'} \operatorname{Re} f(x)\, dx| = \int_{E'} \operatorname{Re} f(x)\, dx \ge \lambda |E'|$. So, $\lambda |E_1| < 2\lambda |E'| \le 2|\langle \operatorname{Re} f, \chi_{E'} \rangle| = 2|\operatorname{Re}\langle f, \chi_{E'} \rangle| \le 2|\langle f, \chi_{E'} \rangle| \le 2A|E_1|^{1-1/p}$. It follows $\lambda^p |E_1| \le (2A)^p$. Do the same for $E_2 = \{\operatorname{Re} f < -\lambda\}$, $E_3 = \{\operatorname{Im} f > \lambda\}$, and $E_4 = \{\operatorname{Im} f < -\lambda\}$ to get $\lambda^p |E_j| \le (2A)^p$ for $j = 1, 2, 3, 4$. But, $F = \{|f| > \lambda\sqrt{2}\} \subseteq \bigcup_j E_j$. So, $(\lambda\sqrt{2})^p |F| \le 4(\sqrt{2})^p (2A)^p$. As λ is arbitrary, we have $\|f\|_{p,\infty} \le 2^{3/2+2/p} A$.

Now suppose $\|f\|_{p,\infty} \le A$. Let $|E| > 0$. Note, $\left|\{|f| > 3^{1/p} A|E|^{-1/p}\}\right| \le \frac{|E|}{3A^p} \|f\|_{p,\infty}^p < |E|/2$. Thus, if $E' = E - \{|f| > 3^{1/p} A|E|^{-1/p}\}$, then $E' \subseteq E$ and $|E'| > |E|/2$. Further, $|\langle f, \chi_{E'} \rangle| \le \int_{E'} |f|\, dm \le |E'| 3^{1/p} A|E|^{-1/p} \lesssim A|E|^{1-1/p}$. □

Theorem 3.10. *For any 0-mean adapted families, $T_\epsilon : L^1 \to L^{1,\infty}$, where the underlying constant is independent of the sequence ϵ.*

Proof. As T_ϵ is linear, it suffices to prove the result for $\|f\|_1 = 1$. Let $|E| > 0$. By Lemma 3.9, we will be done if we can find $E' \subseteq E$, $|E'| > |E|/2$ so that $|\langle T_\epsilon f, \chi_{E'} \rangle| \lesssim 1$. By Theorem 1.10, decompose each ϕ_I^2 into

$$\phi_I^2 = \sum_{k=1}^{\infty} 2^{-10k} \phi_I^{2,k}$$

where $\phi_I^{2,k}$ is the normalization of a 0-mean adapted family $\varphi_I^{2,k}$, which are universally adapted to I. Further, $\operatorname{supp}(\phi_I^{2,k}) \subseteq 2^k I$ for k small enough, while $\phi_I^{2,k}$ is identically 0 otherwise. Now write

$$\langle T_\epsilon f, \chi_{E'}\rangle = \sum_{k=1}^{\infty} 2^{-10k} \sum_I \epsilon_I \langle \phi_I^1, f\rangle \langle \phi_I^{2,k}, \chi_{E'}\rangle.$$

Hence, it suffices to show $|\sum \epsilon_I \langle \phi_I^1, f\rangle\langle \phi_I^{2,k}, \chi_{E'}\rangle| \lesssim 2^{3k}$, so long as the underlying constants are independent of k.

Denote by S^1, $S^{2,k}$ the square functions associated to the appropriate 0-mean adapted families. For each $k \in \mathbb{N}$, define

$$\Omega_{-3k} = \{S^1 f > C2^{3k}\},$$
$$\widetilde{\Omega}_k = \{M(\chi_{\Omega_{-3k}}) > 1/100\},$$
$$\widetilde{\widetilde{\Omega}}_k = \{M(\chi_{\widetilde{\Omega}_k}) > 2^{-k-1}\}.$$

and

$$\Omega = \bigcup_{k\in\mathbb{N}} \widetilde{\widetilde{\Omega}}_k.$$

Observe,

$$|\Omega| \le \sum_{k=1}^{\infty} 2^{-3k} 2^{k+1} \frac{100}{C} \|M\|^2_{L^1\to L^{1,\infty}} \|S^1\|_{L^1\to L^{1,\infty}}.$$

Therefore, we can choose C independent of f so that $|\Omega| < |E|/2$. Set $E' = E - \Omega = E \cap \Omega^c$. Then, $E' \subseteq E$ and $|E'| > |E|/2$.

Fix $k \in \mathbb{N}$. Set $Z_k = \{S^1 f = 0\} \cup \{S^{2,k}(\chi_{E'}) = 0\}$. Let $\mathcal{D}$ be any finite collection of dyadic intervals. We divide this collection into three subcollections. First, define $\mathcal{D}_1 = \{I \in \mathcal{D} : I \cap Z_k \neq \emptyset\}$. For the remaining intervals, let $\mathcal{D}_2 = \{I \in \mathcal{D} - \mathcal{D}_1 : I \subseteq \widetilde{\Omega}_k\}$ and $\mathcal{D}_3 = \{I \in \mathcal{D} - \mathcal{D}_1 : I \cap \widetilde{\Omega}_k^c \neq \emptyset\}$.

If $I \in \mathcal{D}_1$, then there is some $x \in I \cap Z_k$. Namely, either $S^1 f(x) = 0$ or $S^{2,k}(\chi_{E'})(x) = 0$. If it is the first, then from the definition of $S^1 f$, it must be that

$\langle \phi_J^1, f\rangle = 0$ for all dyadic J containing x. In particular, $\langle \phi_I^1, f\rangle = 0$. If instead $S^{2,k}(\chi_{E'})(x) = 0$, then $\langle \phi_I^{2,k}, \chi_{E'}\rangle = 0$. As this holds for all $I \in \mathcal{D}_1$, we have

$$\sum_{I\in\mathcal{D}_1} |\langle \phi_I^1, f\rangle||\langle \phi_I^{2,k}, \chi_{E'}\rangle| = 0.$$

Now suppose $I \in \mathcal{D}_2$, namely $I \subseteq \widetilde{\Omega}_k$. If k is big enough so that $2^k > 1/|I|$, then $\phi_I^{2,k}$ is identically 0 and $\langle \phi_I^{2,k}, \chi_{E'}\rangle = 0$. If $2^k \le 1/|I|$, then $\phi_I^{2,k}$ is supported in $2^k I$. Let $x \in 2^k I$, and observe

$$M(\chi_{\widetilde{\Omega}_k})(x) \ge \frac{1}{|2^k I|}\int_{2^k I} \chi_{\widetilde{\Omega}_k}\, dm \ge \frac{1}{2^k}\frac{1}{|I|}\int_I \chi_{\widetilde{\Omega}_k}\, dm = 2^{-k} > 2^{-k-1}.$$

That is, $2^k I \subseteq \widetilde{\widetilde{\Omega}}_k \subseteq \Omega$, a set disjoint from E'. Thus, $\langle \phi_I^{2,k}, \chi_{E'}\rangle = 0$. As this holds for all $I \in \mathcal{D}_2$, we have

$$\sum_{I\in\mathcal{D}_2} |\langle \phi_I^1, f\rangle||\langle \phi_I^{2,k}, \chi_{E'}\rangle| = 0.$$

Finally, we concentrate on $\mathcal{D}_3$. Define Ω_{-3k+1} and Π_{-3k+1} by

$$\Omega_{-3k+1} = \{S^1 f > C2^{3k-1}\},$$
$$\Pi_{-3k+1} = \{I \in \mathcal{D}_3 : |I \cap \Omega_{-3k+1}| > |I|/100\}.$$

Inductively, define for all $n > -3k+1$,

$$\Omega_n = \{S^1 f > C2^{-n}\},$$
$$\Pi_n = \{I \in \mathcal{D}_3 - \bigcup_{j=-3k+1}^{n-1} \Pi_j : |I \cap \Omega_n| > |I|/100\}.$$

As every $I \in \mathcal{D}_3$ is not in $\mathcal{D}_1$, that is $S^1 f > 0$ on I, it is clear that each $I \in \mathcal{D}_3$ will be in one of these collections.

We can choose an integer N big enough so that $\Omega'_{-N} = \{S^{2,k}(\chi_{E'}) > 2^N\}$ has very small measure. In particular, we take N big enough so that $|I \cap \Omega'_{-N}| < |I|/100$ for all $I \in \mathcal{D}_3$, which is possible since $\mathcal{D}_3$ is a finite collection. Define

$$\Omega'_{-N+1} = \{S^{2,k}(\chi_{E'}) > 2^{N-1}\},$$
$$\Pi'_{-N+1} = \{I \in \mathcal{D}_3 : |I \cap \Omega'_{-N+1}| > |I|/100\},$$

and

$$\Omega'_n = \{S^{2,k}(\chi_{E'}) > 2^{-n}\},$$
$$\Pi'_n = \{I \in \mathcal{D}_3 - \bigcup_{j=-N+1}^{n-1} \Pi'_j : |I \cap \Omega'_n| > |I|/100\},$$

Again, all $I \in \mathcal{D}_3$ must be in one of these collections.

Consider $I \in \mathcal{D}_3$, so that $I \cap \widetilde{\Omega}_k^c \neq \emptyset$. Then, there is some $x \in I \cap \widetilde{\Omega}_k^c$ which implies $|I \cap \Omega_{-3k}|/|I| \leq M(\chi_{\Omega_{-3k}})(x) \leq 1/100$. Write $\Pi_{n_1,n_2} = \Pi_{n_1} \cap \Pi'_{n_2}$. So,

$$\begin{aligned}\sum_{I \in \mathcal{D}_3} |\langle \phi_I^1, f\rangle||\langle \phi_I^{2,k}, \chi_{E'}\rangle| &= \sum_{n_1 > -3k,\, n_2 > -N} \Big[\sum_{I \in \Pi_{n_1,n_2}} |\langle \phi_I^1, f\rangle||\langle \phi_I^{2,k}, \chi_{E'}\rangle| \Big] \\ &= \sum_{n_1 > -3k,\, n_2 > -N} \Big[\sum_{I \in \Pi_{n_1,n_2}} \frac{|\langle \phi_I^1, f\rangle|}{|I|^{1/2}} \frac{|\langle \phi_I^{2,k}, \chi_{E'}\rangle|}{|I|^{1/2}} |I| \Big].\end{aligned}$$

Suppose $I \in \Pi_{n_1,n_2}$. If $n_1 > -3k+1$, then $I \in \Pi_{n_1}$, which in particular says $I \notin \Pi_{n_1-1}$. So, $|I \cap \Omega_{n_1-1}| \leq |I|/100$. If $n_1 = -3k+1$, then we still have $|I \cap \Omega_{-3k}| \leq |I|/100$, as $I \in \mathcal{D}_3$. Similarly, if $n_2 > -N+1$, then $I \notin \Pi'_{n_2-1}$ and $|I \cap \Omega'_{n_2-1}| \leq |I|/100$. If $n_2 = -N+1$, then $|I \cap \Omega'_{-N}| \leq |I|/100$ by the choice of N. So, $|I \cap \Omega^c_{n_1-1} \cap \Omega'^c_{n_2-1}| \geq \frac{98}{100}|I|$. Let $\Omega_{n_1,n_2} = \bigcup\{I : I \in \Pi_{n_1,n_2}\}$. Then, $|I \cap \Omega^c_{n_1-1} \cap \Omega'^c_{n_2-1} \cap \Omega_{n_1,n_2}| \geq \frac{98}{100}|I|$ for all $I \in \Pi_{n_1,n_2}$, and

$$
\begin{aligned}
\sum_{I\in\Pi_{n_1,n_2}} & \frac{|\langle\phi_I^1,f\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^{2,k},\chi_{E'}\rangle|}{|I|^{1/2}}|I| \\
&\lesssim \sum_{I\in\Pi_{n_1,n_2}} \frac{|\langle\phi_I^1,f\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^{2,k},\chi_{E'}\rangle|}{|I|^{1/2}}|I\cap\Omega^c_{n_1-1}\cap\Omega'^c_{n_2-1}\cap\Omega_{n_1,n_2}| \\
&= \int_{\Omega^c_{n_1-1}\cap\Omega'^c_{n_2-1}\cap\Omega_{n_1,n_2}} \sum_{I\in\Pi_{n_1,n_2}} \frac{|\langle\phi_I^1,f\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^{2,k},\chi_{E'}\rangle|}{|I|^{1/2}}\chi_I(x)\,dx \\
&\le \int_{\Omega^c_{n_1-1}\cap\Omega'^c_{n_2-1}\cap\Omega_{n_1,n_2}} S^1f(x)S^{2,k}(\chi_{E'})(x)\,dx \\
&\lesssim C2^{-n_1}2^{-n_2}|\Omega_{n_1,n_2}|.
\end{aligned}
$$

Observe that $|\Omega_{n_1,n_2}| \le |\bigcup\{I : I\in\Pi_{n_1}\}| \le |\{M(\chi_{\Omega_{n_1}}) > 1/100\}| \lesssim |\Omega_{n_1}| = |\{S^1f > C2^{-n_1}\}| \lesssim 2^{n_1}/C$. By the same argument, $|\Omega_{n_1,n_2}| \lesssim |\Omega'_{n_2}| = |\{S^{2,k}(\chi_{E'}) > 2^{-n_2}\}| \lesssim 2^{\alpha n_2}$ for $\alpha = 1,2$, as $S : L^p \to L^{p,\infty}$ for $p = 1,2$. Thus, $|\Omega_{n_1,n_2}| \lesssim C^{-1}2^{\theta_1 n_1}2^{\theta_2\alpha n_2}$ for any $\theta_1+\theta_2 = 1$, $0\le\theta_1,\theta_2\le 1$. Hence,

$$
\begin{aligned}
\sum_{I\in\mathcal{D}_3}|\langle\phi_I^1,f\rangle||\langle\phi_I^{2,k},\chi_{E'}\rangle| &\lesssim \sum_{n_1>-3k,\ n_2>0} 2^{n_1(\theta_1-1)}2^{n_2(\theta_2\alpha-1)} \quad + \\
&\quad \sum_{n_1>-3k,\ -N<n_2\le 0} 2^{n_1(\theta_1-1)}2^{n_2(\theta_2\alpha-1)} \\
&= A+B.
\end{aligned}
$$

For the first term, take $\alpha = 1$, $\theta_1 = \theta_2 = 1/2$, and for the second term, take $\alpha = 2$, $\theta_1 = 1/4$, and $\theta_2 = 3/4$ to see

$$
A = \sum_{n_1>-3k,\ n_2>0} 2^{-n_1/2}2^{-n_2/2} \lesssim 2^{3k/2} \le 2^{3k},
$$

$$
B = \sum_{n_1>-3k,\ -N<n_2\le 0} 2^{-3n_1/4}2^{n_2/2} \le \sum_{n_1=-3k}^{\infty}\sum_{n_2\le 0} 2^{-3n_1/4}2^{n_2/2} \lesssim 2^{9k/4} \le 2^{3k}.
$$

The important thing to notice is that there is no dependence on the number N, which depends on $\mathcal{D}$, or C, which depends on E.

Combining the estimates for $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, we see

$$\sum_{I\in\mathcal{D}} |\langle \phi_I^1, f\rangle||\langle \phi_I^{2,k}, \chi_{E'}\rangle| \lesssim 2^{3k},$$

where the constant has no dependence on the collection $\mathcal{D}$. Hence, as $\mathcal{D}$ is arbitrary, we have

$$\Big|\sum_I \epsilon_I \langle \phi_I^1, f\rangle\langle \phi_I^{2,k}, \chi_{E'}\rangle\Big| \leq \sum_I |\langle \phi_I^1, f\rangle||\langle \phi_I^{2,k}, \chi_{E'}\rangle| \lesssim 2^{3k},$$

which completes the proof. □

Theorem 3.11. *For any 0-mean adapted families, $T_\epsilon : L^p \to L^p$ for $1 < p < \infty$, where the underlying constants are independent of the sequence ϵ.*

Proof. Fix a sequence ϵ_I, and let φ_I^1, φ_I^2 be any two 0-mean adapted families. By Theorems 3.8 and 3.10, $T_\epsilon : L^2 \to L^2$ and $T_\epsilon : L^1 \to L^{1,\infty}$. By the Marcinkiewicz interpolation theorem, $T_\epsilon : L^p \to L^p$ for all $1 < p \leq 2$. By symmetry, the operator $T_\epsilon^* f = \sum \langle \phi_I^2, f\rangle \phi_I^1$ satisfies the same properties.

Fix $f \in L^p$ with $2 < p < \infty$. Let $\|g\|_{p'} \leq 1$, where $1/p + 1/p' = 1$ and $1 < p' < 2$. Then,

$$|\langle T_\epsilon f, g\rangle| = |\langle T_\epsilon^* g, f\rangle| \leq \|T_\epsilon^* g\|_{p'} \|f\|_p \lesssim \|g\|_{p'} \|f\|_p \leq \|f\|_p.$$

As g in the unit ball of $L^{p'}$ is arbitrary, we see $\|T_\epsilon f\|_p \lesssim \|f\|_p$. □

3.4 The L^p Estimates

The main tool is this section is a randomization argument using Khinchtine's inequality. Given a probability space (Ω, P), we say a random variable $r : \Omega \to \mathbb{C}$ is a Rademacher function if $P(r = 1) = P(r = -1) = 1/2$. For more background information on probability spaces and independence, see [2].

Lemma 3.12. *Let $r_1, \ldots, r_N$ be an independent sequence of Rademacher functions on (Ω, P). For any $t > 0$ and any $(a_1, \ldots, a_N) \in \mathbb{C}$ such that $\sum_{j=1}^N |a_j|^2 \leq 1$,*

$$P\Big\{ \Big| \sum_{j=1}^N a_j r_j \Big| > t \Big\} \leq 4e^{-t^2/4}.$$

Proof. First, suppose the a_j are real. Write $S_N(\omega) = \sum_{j=1}^N a_j r_j(\omega)$. We will use the notation $E(\cdot)$ for expectation; that is, $E(f) = \int_\Omega f \, dP$. Recall, if f and g are independent, $E(fg) = E(f)E(g)$. So,

$$E(e^{tS_N}) = E\Big(\prod_{j=1}^N e^{ta_j r_j} \Big) = \prod_{j=1}^N E(e^{ta_j r_j}) = \prod_{j=1}^N \frac{e^{ta_j} + e^{-ta_j}}{2} = \prod_{j=1}^N \cosh(ta_j).$$

Observe $\cosh(x) \leq e^{x^2/2}$ for all real x. So, $E(e^{tS_n}) \leq \prod e^{t^2 a_j^2/2} \leq e^{t^2/2}$. On the other hand,

$$E(e^{tS_N}) \geq \int_{\{S_N > t\}} e^{tS_N(\omega)} \, P(d\omega) \geq e^{t^2} P\{S_N > t\},$$

which implies $P\{S_N > t\} \leq e^{-t^2} E(e^{tS_N}) \leq e^{-t^2/2}$.

Alternatively, $\{S_N < -t\} = \{-S_N > t\}$, where $-S_N = \sum a_j(-r_j)$. As $-r_j$ is also an independent Rademacher sequence, the same applies to $-S_N$. In particular, $P\{-S_N > t\} \leq e^{-t^2/2}$, which gives $P\{|S_N| > t\} \leq P\{S_N > t\} + P\{S_N < -t\} \leq 2e^{-t^2/2}$.

Now allow a_j to be complex with $\sum |a_j|^2 \leq 1$, from which it follows that $\sum |\operatorname{Re} a_j|^2, \sum |\operatorname{Im} a_j|^2 \leq 1$. Let S_N be as before, with $S_N' = \sum \operatorname{Re}(a_j) r_j$ and $S_N'' = \sum \operatorname{Im}(a_j) r_j$. The above argument works with S_N' and S_N'', and therefore $P\{|S_N| > t\} \leq P\{|S_N'| > t/\sqrt{2}\} + P\{|S_N''| > t/\sqrt{2}\} \leq 4e^{-t^2/4}$. □

Theorem 3.13. *For each $0 < p < \infty$, any sequence of complex numbers $\{a_j\}_{j \in \mathbb{N}}$ in ℓ^2, and any independent sequence of Rademacher functions $\{r_j\}$ on Ω, we have*

$$\left(\sum_{j=1}^{\infty}|a_j|^2\right)^{1/2} \sim \left\|\sum_{j=1}^{\infty}a_jr_j\right\|_{L^p(\Omega)},$$

where the underlying constants depend only on p.

Proof. Fix $N \in \mathbb{N}$. Write $\sigma^2 = \sum_{j=1}^{N}|a_j|^2$ and define $b_j = a_j/\sigma$, so that $\sum_{j=1}^{N}|b_j|^2 = 1$. Let $S_N = \sum_{j=1}^{N} b_jr_j$. Then, using the previous lemma,

$$\int_{\Omega}|S_N(\omega)|^p\,P(d\omega) = \int_0^{\infty} pt^{p-1}P\{|S_N| > t\}\,dt \leq 4p\int_0^{\infty} t^{p-1}e^{-t^2/4}\,dt =: K_p^p,$$

where $K_p < \infty$ for all $0 < p < \infty$.

Suppose $1 < p < \infty$. Note, by independence, $E(r_jr_k) = E(r_j)E(r_k) = 0$ for $j \neq k$ and $E(r_jr_j) = 1$. So,

$$\int_{\Omega}|S_N|^2\,dP = \int_{\Omega}S_N\overline{S_N}\,dP = \sum_{1\leq j,k\leq N} b_j\overline{b_k}\int_{\Omega}r_jr_k\,dP = \sum_{j=1}^{N}|b_j|^2 = 1.$$

But, by above, $\|S_N\|_{p'} \leq K_{p'}$. So, by Hölder, $1 \leq \|S_N\|_p\|S_N\|_{p'} \leq \|S_N\|_pK_{p'}$, or $K_{p'}^{-1} \leq \|S_N\|_p$. Now suppose $0 < p \leq 1$. Then, $1 = \int_{\Omega}|S_N|^2\,dP \leq \||S_N|^{p/2}\|_2\||S_N|^{2-p/2}\|_2 = \|S_N\|_p^{2/p}\|S_N\|_{4-p}^{2/(4-p)}$. Note, $\|S_N\|_{4-p}^{2/(4-p)} \leq K_{4-p}^{2/(4-p)}$. Therefore, $K_{4-p}^{p/(p-4)} \leq \|S_N\|_p$. Let $K_p' = K_{p'}^{-1}$ for $p > 1$ and $K_p' = K_{4-p}^{p/(4-p)}$ for $p \leq 1$. Then, we have shown

$$K_p'\left(\sum_{j=1}^{N}|a_j|^2\right)^{1/2} \leq \left\|\sum_{j=1}^{N}a_jr_j\right\|_{L^p(\Omega)} \leq K_p\left(\sum_{j=1}^{N}|a_j|^2\right)^{1/2}.$$

for all $0 < p < \infty$. To finish, we note that by Fatou's Lemma

$$\int_{\Omega}\Big|\sum_{j=1}^{\infty}a_jr_j\,dP\Big|^p \leq \liminf_{N\to\infty}\int_{\Omega}\Big|\sum_{j=1}^{N}a_jr_j\,dP\Big|^p \leq K_p\left(\sum_{j=1}^{\infty}|a_j|^2\right)^{p/2}.$$

Fix $1 \leq p < \infty$. Then, by Minkowski,

$$\Big\|\sum_{j=1}^{N} a_j r_j\Big\|_{L^p(\Omega)} - \Big\|\sum_{j=1}^{\infty} a_j r_j\Big\|_{L^p(\Omega)} \le \Big\|\sum_{j=N+1}^{\infty} a_j r_j\Big\|_{L^p(\Omega)} \le K_p\Big(\sum_{j=N+1}^{\infty} |a_j|^2\Big)^{1/2},$$

the last term tending to 0 as $N \to \infty$, because (a_j) is in ℓ^2. Thus,

$$\Big\|\sum_{j=1}^{\infty} a_j r_j\Big\|_{L^p(\Omega)} \ge \limsup_{N\to\infty}\Big\|\sum_{j=1}^{N} a_j r_j\Big\|_{L^p(\Omega)} \ge K'_p\Big(\sum_{j=1}^{\infty} |a_j|^2\Big)^{1/2}.$$

Finally, let $0 < p < 1$. Set $t = (2-2p)/(2-p)$ so that $0 < t < 1$ and $1 = (1-t)/p + t/2$. Let $F = \sum_{j=1}^{\infty} a_j r_j$. Then,

$$\begin{aligned}\|F\|_{L^1(\Omega)} &= \||F|^{1-t}|F|^t\|_{L^1(\Omega)} \le \||F|^{1-t}\|_{L^{p/(1-t)}(\Omega)}\||F|^t\|_{L^{2/t}(\Omega)} = \|F\|^{1-t}_{L^p(\Omega)}\|F\|^t_{L^2(\Omega)}\\ &\le \|F\|^{1-t}_{L^p(\Omega)}(K_2\|a\|_{\ell^2})^t \le \|F\|^{1-t}_{L^p(\Omega)}(K_2K_1'^{-1}\|F\|_{L^1(\Omega)})^t\end{aligned}$$

which implies $K'_1\|a\|_{\ell^2} \le \|F\|_{L^1(\Omega)} \le (K_2/K'_1)^{t/(1-t)}\|F\|_{L^p(\Omega)}$, completing the proof. □

Theorem 3.14. *For any 0-mean adapted family, $S : L^p \to L^p$ for $1 < p < \infty$.*

Proof. Let φ_I be a 0-mean adapted family and S the associated square operator. By Theorem 1.7, let φ_I^2 be a second adapted family, with the additional property that $\chi_I \lesssim |\varphi_I^2|$ for all I. That is, $\chi_I/|I|^{1/2} \lesssim |\phi_I^2|$.

Let $\{r_I\}$ be an independent sequence of Rademacher functions on a probability space (Ω, P) indexed by the dyadic intervals. For each $\omega \in \Omega$, denote the sequence $\{r_I(\omega)\}$ by $\epsilon(\omega)_I$, and note $|\epsilon(\omega)_I| \le 1$ for all I. Let $T_{\epsilon(\omega)}$ be the linearization associated to φ_I, φ_I^2, and the sequence $\epsilon(\omega)$.

Fix $1 < p < \infty$ and $f \in L^p$. By Khinctine,

$$\begin{aligned}|Sf(x)|^p &= \Big(\sum_I \frac{|\langle\phi_I, f\rangle|^2}{|I|}\chi_I(x)\Big)^{p/2} \lesssim \Big(\sum_I |\langle\phi_I, f\rangle|^2|\phi_I^2(x)|^2\Big)^{p/2}\\ &\lesssim \int_\Omega \Big|\sum_I r_I(\omega)\langle\phi_I, f\rangle\phi_I^2(x)\Big|^p P(d\omega) = \int_\Omega |T_{\epsilon(\omega)}f(x)|^p\, P(d\omega).\end{aligned}$$

So,

$$\begin{aligned}\|Sf\|_p^p = \int_{\mathbb{T}} |Sf(x)|^p \, dx &\lesssim \int_{\mathbb{T}} \int_{\Omega} |T_{\epsilon(\omega)} f(x)|^p \, dx \, P(d\omega) \\ &= \int_{\Omega} \left\|T_{\epsilon(\omega)} f\right\|_p^p P(d\omega) \lesssim \int_{\Omega} \|f\|_p^p \, P(d\omega) = \|f\|_p^p.\end{aligned}$$

□

3.5 Fefferman-Stein Inequalities

We are also able to prove a special case of Fefferman-Stein inequalities ($r = 2$) for the square function. First, we need the following characterization of weak-L^p, sometimes called the Kolmogorov condition.

Lemma 3.15. *Let $0 < r < p < \infty$, and choose s so that $1/s = 1/r - 1/p$. Denote*

$$M_{p,r}(f) = \sup \left\{ \frac{\|f\chi_E\|_r}{\|\chi_E\|_s} : |E| > 0 \right\}.$$

Then, $\|f\|_{p,\infty} \sim M_{p,r}(f)$ for all f, where the underlying constant depends only on p and r.

Proof. Let $\lambda > 0$ and $E = \{|f| > \lambda\}$. If $|E| = 0$, then $\lambda |E|^{1/p} \leq M_{p,r}(f)$ trivially. So, assume $|E| > 0$. Then,

$$|E|^{1/r} = \Big(\int_E dx \Big)^{1/r} \leq \lambda^{-1} \Big(\int_E |f(x)|^r \, dx \Big)^{1/r} = \lambda^{-1} \|f\chi_E\|_r \leq \lambda^{-1} \|\chi_E\|_s M_{p,r}(f).$$

Hence, $M_{p,r}(f) \geq \lambda |E|^{1/r - 1/s} = \lambda |E|^{1/p}$. As λ is arbitrary, $\|f\|_{p,\infty} \leq M_{p,r}(f)$.

If $\|f\|_{p,\infty} = \infty$, the reverse inequality is trivially satisfied. So, assume it is finite. If $\|f\|_{p,\infty} = 0$, then $f = 0$ a.e., and again the reverse inequality holds. Assume $\|f\|_{p,\infty} > 0$. Set $g = f/\|f\|_{p,\infty}$ which gives $\|g\|_{p,\infty} = 1$. Let $|E| > 0$. Then, $|\{|g\chi_E| > \lambda\}| \leq \min(|E|, \lambda^{-p})$. Thus, for any $h > 0$

$$\begin{aligned}\|g\chi_E\|_r^r &= \int_0^\infty r\lambda^{r-1}\big|\{|g\chi_E| > \lambda\}\big|\,d\lambda \\ &\le r|E|\int_0^h \lambda^{r-1}\,d\lambda + r\int_h^\infty \lambda^{r-p-1}\,d\lambda \\ &= h^r|E| + \frac{r}{p-r}h^{r-p}.\end{aligned}$$

Setting $h = |E|^{-1/p}$ implies $\|g\chi_E\|_r^r \le |E|^{r/s} + \frac{r}{p-r}|E|^{r/s}$ and $\|g\chi_E\|_r \le (\frac{p}{p-r})^{1/r}|E|^{1/s} = (\frac{p}{p-r})^{1/r}\|\chi_E\|_s$. As E is arbitrary, $M_{p,r}(g) \le (\frac{p}{p-r})^{1/r}$. Noting that $M_{p,r}$ is quasi-linear, we have $M_{p,r}(f) \le (\frac{p}{p-r})^{1/r}\|f\|_{p,\infty}$. □

Theorem 3.16. *For $1 < p < \infty$ and any sequence $f_1, f_2, \dots$ of complex-valued functions on $\mathbb{T}$*

$$\Big\|\Big(\sum_{k=1}^\infty |Sf_k|^2\Big)^{1/2}\Big\|_p \lesssim \Big\|\Big(\sum_{k=1}^\infty |f_k|^2\Big)^{1/2}\Big\|_p,$$
$$\Big\|\Big(\sum_{k=1}^\infty |Sf_k|^2\Big)^{1/2}\Big\|_{1,\infty} \lesssim \Big\|\Big(\sum_{k=1}^\infty |f_k|^2\Big)^{1/2}\Big\|_1.$$

Proof. Let r_I be a sequence of independent Rademacher functions, indexed by the dyadic intervals, on a probability space (Ω, P). Let r'_k be another sequence of independent Rademacher functions, indexed by $\mathbb{N}$, on a probability space (Ω', P'). Note, $r_{I,k}(\omega,\omega') = r_I(\omega)r'_k(\omega')$ is an independent Rademacher sequence on $\Omega\times\Omega'$.

Let $1 < p < \infty$. Fix $N \in \mathbb{N}$. Then, by Khinchtine,

$$\begin{aligned}&\Big\|\Big(\sum_{k=1}^N |Sf_k|^2\Big)^{1/2}\Big\|_p^p \\ &\quad= \int_{\mathbb{T}}\Big(\sum_{k=1}^N\sum_I \frac{|\langle\phi_I, f_k\rangle|^2}{|I|}\chi_I(x)\Big)^{p/2}dx \\ &\quad\lesssim \int_{\mathbb{T}}\int_{\Omega\times\Omega'}\Big|\sum_{k=1}^N\sum_I r_I(\omega)r'_k(\omega')\frac{1}{|I|^{1/2}}\langle\phi_I, f_k\rangle\chi_I(x)\Big|^p P(d\omega)\,P(d\omega')\,dx \\ &\quad= \int_{\mathbb{T}}\int_{\Omega\times\Omega'}\Big|r_I(\omega)\frac{1}{|I|^{1/2}}\Big\langle\phi_I, \sum_{k=1}^N r'_k(\omega')f_k\Big\rangle\chi_I(x)\Big|^p P(d\omega)\,P'(d\omega')\,dx.\end{aligned}$$

Now use the reverse inequality of Khinctine, first in Ω, then Ω', to see

$$\begin{aligned}
\Big\| \Big(\sum_{k=1}^{N} |Sf_k|^2 \Big)^{1/2} \Big\|_p^p &\lesssim \int_{\Omega'} \int_{\mathbb{T}} \Big(\sum_I \frac{1}{|I|} \Big| \Big\langle \phi_I, \sum_{k=1}^{N} r_k'(\omega') f_k \Big\rangle \Big|^2 \chi_I(x) \Big)^{p/2} \, dx \, P'(d\omega') \\
&= \int_{\Omega'} \int_{\mathbb{T}} \Big| S\Big(\sum_{k=1}^{N} r_k'(\omega') f_k \Big)(x) \Big|^p \, dx \, P'(d\omega') \\
&\lesssim \int_{\Omega'} \int_{\mathbb{T}} \Big| \sum_{k=1}^{N} r_k'(\omega') f_k(x) \Big|^p \, dx \, P'(d\omega') \\
&\lesssim \int_{\mathbb{T}} \Big(\sum_{k=1}^{N} |f_k(x)|^2 \Big)^{p/2} \, dx = \Big\| \Big(\sum_{k=1}^{N} |f_k|^2 \Big)^{1/2} \Big\|_p^p.
\end{aligned}$$

Simply apply the monotone convergence theorem to let $N \to \infty$ and gain the desired result.

Now let $|E| > 0$. Fix $0 < r < 1$ and $1/s = 1/r - 1$. As $\|Sf\|_{1,\infty} \lesssim \|f\|_1$, it follows from Lemma 3.15 that $\|S(f)\chi_E\|_r \lesssim \|\chi_E\|_s \|f\|_1$. Again, fix $N \in \mathbb{N}$. So,

$$\begin{aligned}
&\Big\| \Big(\sum_{k=1}^{N} |Sf_k|^2 \Big)^{1/2} \chi_E \Big\|_r^r \\
&= \int_{\mathbb{T}} \Big(\sum_{k=1}^{N} \sum_I \frac{|\langle \phi_I, f_k \rangle|^2}{|I|} \chi_I(x) \chi_E(x) \Big)^{r/2} \, dx \\
&\lesssim \int_{\mathbb{T}} \int_{\Omega \times \Omega'} \Big| \sum_{k=1}^{N} \sum_I r_I(\omega) r_k'(\omega') \frac{1}{|I|^{1/2}} \langle \phi_I, f_k \rangle \chi_I(x) \chi_E(x) \Big|^r \, P(d\omega) \, P'(d\omega') \, dx \\
&= \int_{\mathbb{T}} \int_{\Omega \times \Omega'} \Big| r_I(\omega) \frac{1}{|I|^{1/2}} \Big\langle \phi_I, \sum_{k=1}^{N} r_k'(\omega') f_k \Big\rangle \chi_I(x) \chi_E(x) \Big|^r \, P(d\omega) \, P'(d\omega') \, dx \\
&\lesssim \int_{\Omega'} \int_{\mathbb{T}} \Big| S\Big(\sum_{k=1}^{N} r_k'(\omega') f_k \Big)(x) \chi_E(x) \Big|^r \, dx \, P'(d\omega') \\
&\lesssim \|\chi_E\|_s^r \int_{\Omega'} \Big[\int_{\mathbb{T}} \Big| \sum_{k=1}^{N} r_k'(\omega') f_k(x) \Big| \, dx \Big]^r \, P'(d\omega')
\end{aligned}$$

As Ω' is a probability space and $r < 1$, we can apply Jensen's inequality to see

$$
\begin{aligned}
\Big\|\Big(\sum_{k=1}^{N}|Sf_k|^2\Big)^{1/2}\chi_E\Big\|_r &\lesssim \|\chi_E\|_s\bigg(\int_{\Omega'}\Big[\int_{\mathbb{T}}\Big|\sum_{k=1}^{N}r_k'(\omega')f_k(x)\Big|\,dx\Big]^r P(d\omega')\bigg)^{1/r} \\
&\le \|\chi_E\|_s\int_{\Omega'}\int_{\mathbb{T}}\Big|\sum_{k=1}^{N}r_k'(\omega')f_k(x)\Big|\,dx\,P(d\omega') \\
&\lesssim \|\chi_E\|_s^r\int_{\mathbb{T}}\Big(\sum_{k=1}^{N}|f_k(x)|^2\Big)^{r/2}dx \\
&= \|\chi_E\|_s^r\Big\|\Big(\sum_{k=1}^{N}|f_k|^2\Big)^{1/2}\Big\|_r^r.
\end{aligned}
$$

Taking the supremum over all such E, and applying Lemma 3.15,

$$
\Big\|\Big(\sum_{k=1}^{N}|Sf_k|^2\Big)^{1/2}\Big\|_{1,\infty} \lesssim M_{1,r}\bigg(\Big(\sum_{k=1}^{N}|Sf_k|^2\Big)^{1/2}\bigg) \lesssim \Big\|\Big(\sum_{k=1}^{N}|f_k|^2\Big)^{1/2}\Big\|_1.
$$

Letting $N\to\infty$ completes the proof. $\square$

Chapter 4
Zygmund Spaces and $L \log L$

In this chapter, we begin by focusing on a general measure space (X, ρ). Our goal is to introduce new spaces of functions and interpolation results that will ultimately give us the "end-point" estimates of certain operators. Many of the preliminary proofs of this chapter are taken from [1].

4.1 Decreasing Rearrangements

Definition. For $f : (X, \rho) \to \mathbb{C}$, the distribution function of f is defined

$$\mu_f(\lambda) = \rho\{x \in X : |f(x)| > \lambda\}, \quad \lambda \geq 0.$$

Two function f, g (even if they act on different measure spaces) are said to be equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$.

Definition. For $f : (X, \rho) \to \mathbb{C}$, the deceasing rearrangement of f is defined

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad t \geq 0,$$

where we use the convention that $\inf\{\emptyset\} = \infty$.

Note, if (X, ρ) is a finite measure space, then $\mu_f(\lambda) \leq \rho(X)$ for all $\lambda \geq 0$. Hence, $f^*(t) = 0$ for all $t > \rho(X)$. That is, f^* is supported in $[0, \rho(X)]$.

Proposition 4.1. *For any $f, f_n, g : (X, \rho) \to \mathbb{C}$ and $\alpha \in \mathbb{C}$,*

1. *f^* is nonnegative, decreasing, and identically 0 if and only if $f = 0$ a.e.[ρ],*
2. *$|f| \leq |g|$ a.e.[ρ] implies $f^* \leq g^*$ pointwise,*

3. $f^*(\mu_f(\lambda)) \leq \lambda$ *for* $\mu_f(\lambda) < \infty$*, and* $\mu_f(f^*(t)) \leq t$ *for* $f^*(t) < \infty$*,*

4. $(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2)$*,*

5. $(\alpha f)^* = |\alpha| f^*$*,*

6. $|f_n| \uparrow |f|$ *a.e.[*ρ*] implies* $f_n^* \uparrow f^*$ *pointwise,*

7. f *and* f^* *are equimeasurable.*

Proof. (1) The fact that $f^* \geq 0$ follows from the definition. Let $0 \leq t_1 < t_2$ and $\epsilon > 0$. Choose $\lambda \geq 0$ so that $\mu_f(\lambda) \leq t_1$ and $f^*(t_1) + \epsilon \geq \lambda$. Then, $\mu_f(\lambda) \leq t_1 < t_2$ which implies $f^*(t_2) \leq \lambda \leq f^*(t_1) + \epsilon$. As ϵ is arbitrary, $f^*(t_1) \geq f^*(t_2)$. But, since f^* is decreasing, f^* is identically 0 if and only if $f^*(0) = 0$. This is true if and only if $\mu_f(0) = 0$, which means $f = 0$ a.e..

(2) Fix t and $\epsilon > 0$. As $|f| \leq |g|$ a.e., it is immediately clear that $\mu_f \leq \mu_g$. Choose $\lambda \geq 0$ so that $\mu_g(\lambda) \leq t$ and $g^*(t) + \epsilon \geq \lambda$. Then, $\mu_f(\lambda) \leq \mu_g(\lambda) \leq t$, which implies that $f^*(t) \leq \lambda \leq g^*(t) + \epsilon$. As ϵ is arbitrary, $f^*(t) \leq g^*(t)$.

(3) Fix $\lambda \geq 0$ and set $t = \mu_f(\lambda)$. Then, $\lambda \in \{\lambda' \geq 0 : \mu_f(\lambda') \leq t\}$ giving $f^*(\mu_f(\lambda)) = f^*(t) = \inf\{\lambda' : \mu_f(\lambda') \leq t\} \leq \lambda$. Now fix $t \geq 0$ and assume $\lambda = f^*(t) < \infty$. Let λ_n be a sequence of positive numbers so that $\lambda_n \downarrow \lambda$. Then, $\mu_f(\lambda_n) \leq t$ for each n. Therefore, as $\{|f| > \lambda_n\} \subseteq \{|f| > \lambda\}$ for all n and

$$\bigcup_n \{|f| > \lambda_n\} = \{|f| > \lambda\},$$

it follows from simple properties of measures that $\mu_f(\lambda_n) \uparrow \mu_f(\lambda)$. That is, $\mu_f(\lambda) = \lim_n \mu_f(\lambda_n) \leq t$.

(4) Let $t_1, t_2 \geq 0$. Let $\lambda = f^*(t_1) + f^*(t_2)$ and $t = \mu_{f+g}(\lambda)$. Then,

$$\begin{aligned} t &= |\{|f+g| > \lambda\}| \leq |\{|f| > f^*(t_1)\}| + |\{|g| > g^*(t_2)\}| \\ &= \mu_f(f^*(t_1)) + \mu_g(g^*(t_2)) \leq t_1 + t_2. \end{aligned}$$

So, $(f+g)^*(t_1+t_2) \le (f+g)^*(t) = (f+g)^*(\mu_{f+g}(\lambda)) \le \lambda = f^*(t_1) + f^*(t_2)$.

(5) For $\alpha \in \mathbb{C}$, we have $\mu_{\alpha f}(\lambda) = \rho\{|\alpha f| > \lambda\} = \rho\{|f| > \lambda/|\alpha|\} = \mu_f(\lambda/|\alpha|)$. Thus, $(\alpha f)^*(t) = \inf\{\lambda \ge 0 : \mu_{\alpha f}(\lambda) \le t\} = \inf\{|\alpha|\lambda \ge 0 : \mu_f(\lambda) \le t\} = |\alpha| f^*(t)$.

(6) It is clear from (2) that $f_1^* \le f_2^* \le \ldots \le f^*$ pointwise. Fix λ. By the same argument used in (3), we see $\{|f_n| > \lambda\} \subseteq \{|f| > \lambda\}$ and $\bigcup\{|f_n| > \lambda\} = \{|f| > \lambda\}$. Thus, $\mu_{f_n}(\lambda) \uparrow \mu_f(\lambda)$. By the same token, it is now clear that $\{\lambda : \mu_f(\lambda) \le t\} \subseteq \{\lambda : \mu_{f_n}(\lambda) \le t\}$ and $\bigcap_n \{\lambda : \mu_{f_n}(\lambda) \le t\} = \{\lambda : \mu_f(\lambda) \le t\}$. Therefore, taking infimums, we see $f_n^*(t) \uparrow f^*(t)$.

(7) Simply from the definition, $f^*(t) > \lambda$ if and only if $t < \mu_f(\lambda)$. Thus, $\mu_{f^*}(\lambda) = |\{t \ge 0 : f^*(t) > \lambda\}| = |[0, \mu_f(\lambda))| = \mu_f(\lambda)$. $\square$

Lemma 4.2. *Let $\Psi : [0,\infty) \to [0,\infty)$ be continuous and increasing with $\Psi(0) = 0$. Then, $\int_X \Psi(|f|)\, d\rho = \int_0^\infty \Psi(f^*)\, dt$.*

Proof. First consider the case where f is positive and simple. That is, there are constants $a_1 > a_2 > \ldots > a_n > 0$ and disjoint sets $E_1, \ldots, E_n$ so that $f = \sum a_j \chi_{E_j}$. It is easy to calculate that $f^*(t) = \sum a_j \chi_{[m_{j-1}, m_j)}$, where $m_0 = 0$ and $m_j = \rho(E_1) + \ldots + \rho(E_j)$. Thus,

$$\begin{aligned} \int_X \Psi(f)\, d\rho &= \sum_{j=1}^n \int_{E_j} \Psi(a_j)\, d\rho = \sum_{j=1}^n \Psi(a_j)\rho(E_j) = \sum_{j=1}^n \Psi(a_j)\big[m_j - m_{j-1}\big] \\ &= \sum_{j=1}^n \int_0^\infty \Psi(a_j)\chi_{[m_{j-1},m_j)}(t)\, dt = \int_0^\infty \Psi(f^*)\, dt. \end{aligned}$$

Note that $\Psi(0) = 0$ was used here to say $\Psi(a_j\chi_{E_j}) = \Psi(a_j)\chi_{E_j}$.

Now consider a general $f : (X, \rho) \to \mathbb{C}$. Choose positive simple functions f_n so that $f_n \uparrow |f|$. As Ψ is continuous and increasing, it follows that $\Psi(f_n) \uparrow \Psi(|f|)$. Also, as $f_n^* \uparrow f^*$, we have $\Psi(f_n^*) \uparrow \Psi(f^*)$. So, by the monotone convergence theorem,

$$\int_X \Psi(|f|)\,d\rho = \lim_{n\to\infty}\int_X \Psi(f_n)\,d\rho = \lim_{n\to\infty}\int_0^\infty \Psi(f_n^*)\,dt = \int_0^\infty \Psi(f^*)\,dt.$$

□

Corollary 4.3. *For $f : (X,\rho) \to \mathbb{C}$ and $0 < p < \infty$, we have $\|f\|_p = \|f^*\|_p$. Furthermore, $\|f\|_\infty = \|f^*\|_\infty = f^*(0)$.*

Proof. In the case $0 < p < \infty$, simply let $\Psi(t) = t^p$ and apply the previous lemma. Secondly, note that $\|f\|_\infty = f^*(0)$ by definition. As f^* is decreasing, $\|f^*\|_\infty = f^*(0)$. □

4.2 Lorentz Spaces

Definition. Let $0 < p < \infty$ and $0 < q \le \infty$. For $f : (X,\rho) \to \mathbb{C}$, define $\|f\|_{p,q}$ by

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q}, & q < \infty \\ \sup\limits_{t>0} t^{1/p} f^*(t), & q = \infty. \end{cases}$$

Denote by $L^{p,q}(X)$ be the set of functions f for which $\|f\|_{p,q} < \infty$.

It is clear from Corollary 4.3 that $\|f\|_{p,p} = \|f\|_p$. Further, one can check that $L^{p,\infty}$ here coincides with the definition of weak-L^p given in Section 1.2.

Lemma 4.4. *Let $0 < p < \infty$ and $0 < q < r \le \infty$. Then, $\|f\|_{p,r} \lesssim \|f\|_{p,q}$, where the underlying constants depend only on p, q, r.*

Proof. As f^* is decreasing,

$$\begin{aligned} t^{1/p} f^*(t) &= \left(\frac{p}{q}\int_0^t \left(s^{1/p} f^*(t)\right)^q \frac{ds}{s}\right)^{1/q} \\ &\le \left(\frac{p}{q}\int_0^t \left(s^{1/p} f^*(s)\right)^q \frac{ds}{s}\right)^{1/q} = \left(\frac{p}{q}\right)^{1/q} \|f\|_{p,q}. \end{aligned}$$

Taking the supremum over all t, we see $\|f\|_{p,\infty} \leq (\frac{p}{q})^{1/q}\|f\|_{p,q}$. This gives the $r = \infty$ case. Now, suppose $r < \infty$. Then,

$$\begin{aligned}\|f\|_{p,r} &= \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^{r-q+q} \frac{dt}{t}\right)^{1/r} \\ &\leq \|f\|_{p,\infty}^{1-q/r}\|f\|_{p,q}^{q/r} \leq \left(\left(\frac{p}{q}\right)^{1/q}\|f\|_{p,q}\right)^{1-q/r}\|f\|_{p,q}^{q/r} = \left(\frac{p}{q}\right)^{1/q-1/r}\|f\|_{p,q}.\end{aligned}$$

□

Lemma 4.5. *Let T be a sublinear operator which which maps $L^{p_0}(X) \to L^{q_0,\infty}(X)$ and $L^{p_1}(X) \to L^{q_1,\infty}(X)$, where $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1 < \infty$, and $q_0 \neq q_1$. Then,*

$$(Tf)^*(t) \lesssim \left[t^{-1/q_0} \int_0^{t^m} s^{1/p_0} f^*(s)\,\frac{ds}{s} + t^{-1/q_1} \int_{t^m}^\infty s^{1/p_1} f^*(s)\,\frac{ds}{s}\right], \quad t > 0,$$

where $m = (\frac{1}{q_0} - \frac{1}{q_1})(\frac{1}{p_0} - \frac{1}{p_1})^{-1}$.

Proof. Let $\alpha(x)$ be a complex-valued function with $|\alpha(x)| = 1$ so that $|f(x)|\alpha(x) = f(x)$. Fix $t > 0$. Define f_0 and f_1 by

$$\begin{aligned}f_0(x) &= \max\left\{|f(x)| - f^*(t^m), 0\right\} \cdot \alpha(x), \\ f_1(x) &= \min\left\{|f(x)|, f^*(t^m)\right\} \cdot \alpha(x).\end{aligned}$$

Then, $f = f_0 + f_1$, and it is easily shown that $f_0^*(s) = \max\{f^*(s) - f^*(t^m), 0\}$ and $f_1^*(s) = \min\{f^*(s), f^*(t^m)\}$. Further,

$$\begin{aligned}\|f_0\|_{p_0,1} &= \int_0^{t^m} s^{1/p_0} f^*(s)\,\frac{ds}{s} - p_0 t^{m/p_0} f^*(t^m), \\ \|f_1\|_{p_1,1} &= p_1 t^{m/p_1} f^*(t^m) + \int_{t^m}^\infty s^{1/p_1} f^*(s)\,\frac{ds}{s}.\end{aligned}$$

As T is sublinear, we have that $(Tf)^*(t) \leq (Tf_0 + Tf_1)^*(t) \leq (Tf_0)^*(t/2) + (Tf_1)^*(t/2)$. By the hypotheses on T,

$$\left(\frac{t}{2}\right)^{1/q_0}(Tf_0)^*(t/2) \leq \|Tf_0\|_{q_0,\infty} \lesssim \|f_0\|_{p_0} \lesssim \|f_0\|_{p_0,1},$$

or

$$(Tf_0)^*(t/2) \lesssim t^{-1/q_0}\|f_0\|_{p_0,1}.$$

Similarly,

$$(Tf_1)^*(t/2) \lesssim t^{-1/q_1}\|f_1\|_{p_1,1}.$$

Hence,

$$\begin{aligned}
(Tf)^*(t) &\leq (Tf_0)^*(t/2) + (Tf_1)^*(t/2) \\
&\lesssim \left[\frac{1}{p_0}t^{-1/q_0}\|f_0\|_{p_0,1} + \frac{1}{p_1}t^{-1/q_1}\|f_1\|_{p_1,1}\right] \\
&= \Bigg[\frac{1}{p_0}t^{-1/q_0}\int_0^{t^m} s^{1/p_0} f^*(s)\,\frac{ds}{s} + \frac{1}{p_1}t^{-1/q_1}\int_{t^m}^{\infty} s^{1/p_1} f^*(s)\,\frac{ds}{s} \\
&\qquad + t^{m/p_1 - 1/q_1} f^*(t^m) - t^{m/p_0 - 1/q_0} f^*(t^m)\Bigg]
\end{aligned}$$

Noting that $\frac{m}{p_0} - \frac{1}{q_0} = \frac{m}{p_1} - \frac{1}{q_1}$, the $f^*(t^m)$ terms cancel. Thus,

$$(Tf)^*(t) \lesssim \left[t^{-1/q_0}\int_0^{t^m} s^{1/p_0} f^*(s)\,\frac{ds}{s} + t^{-1/q_1}\int_{t^m}^{\infty} s^{1/p_1} f^*(s)\,\frac{ds}{s}\right].$$

□

4.3 The 2-Star Operator

The next step is to define a kind of maximal operator of f^*, which we call the 2-star operator.

Definition. For $f : (X, \rho) \to \mathbb{C}$, define

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds, \quad t > 0.$$

Proposition 4.6. *For any $f, f_n, g : (X, \rho) \to \mathbb{C}$ and $\alpha \in \mathbb{C}$,*

1. *f^{**} is nonnegative, decreasing, and identically 0 if and only if $f = 0$ a.e.[ρ],*
2. *$f^* \leq f^{**}$,*
3. *$|f| \leq |g|$ a.e.[ρ] implies $f^{**} \leq g^{**}$ pointwise,*
4. *$(\alpha f)^{**} = |\alpha| f^{**}$,*
5. *$|f_n| \uparrow |f|$ a.e.[ρ] implies $f_n^{**} \uparrow f^{**}$ pointwise.*

Proof. The fact that f^{**} is nonnegative and equal to 0 if and only if $f = 0$ a.e. follows as f^* satisfies the same properties. Let $0 \leq t_1 < t_2$. As f^* is decreasing $f^*(s) \leq f^*(st_1/t_2)$ for any $s \geq 0$. Thus,

$$f^{**}(t_2) = \frac{1}{t_2} \int_0^{t_2} f^*(s)\, ds \leq \frac{1}{t_2} \int_0^{t_2} f^*(st_1/t_2)\, ds = \frac{1}{t_1} \int_0^{t_1} f^*(u)\, du = f^{**}(t_1).$$

This establishes (1). Again, as f^* is decreasing,

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds \geq f^*(t) \frac{1}{t} \int_0^t dt = f^*(t).$$

This establishes (2). Properties (3), (4), and (5) follow immediately from the fact that f^* satisfies the same properties, in addition to the monotone convergence theorem for (5). $\square$

We will also want to show that the 2-star operator is sublinear. This is more difficult than the preceding results, and needs the following intermediary step.

Lemma 4.7. *For all $t > 0$, $\inf\limits_{f=g+h} \left\{ \|g\|_1 + t\|h\|_\infty \right\} = t f^{**}(t)$.*

Proof. Fix $t > 0$ and $f : (X, \rho) \to \mathbb{C}$. Let α_t be the value of the infimum on the left-hand side of the equality. We first show $tf^{**}(t) \le \alpha_t$.

We can assume that f can be decomposed into $g + h$ as implied, as otherwise $\alpha_t = \infty$ and there is nothing to prove. So, write $f = g + h$ where $g \in L^1(X)$ and $h \in L^\infty(X)$. Let $n \in \mathbb{N}$. Then,

$$\begin{aligned} tf^{**}(t) &= \int_0^t f^*(s)\,ds \le \int_0^t g^*\Big(\frac{n-1}{n}\,s\Big)\,ds + \int_0^t h^*\Big(\frac{1}{n}\,s\Big)\,ds \\ &= \frac{n}{n-1}\int_0^{t(n-1)/n} g^*(u)\,du + n\int_0^{t/n} h^*(u)\,du \\ &\le \frac{n}{n-1}\int_0^{\infty} g^*(u)\,du + nh^*(0)\int_0^{t/n} du \\ &= \frac{n}{n-1}\|g\|_1 + t\|h\|_\infty. \end{aligned}$$

As n is arbitrary, let $n \to \infty$ to see $tf^{**}(t) \le \|g\|_1 + t\|h\|_\infty$. As this decomposition is arbitrary, $tf^{**}(t) \le \alpha_t$.

For the reverse inequality, we can assume $f^{**}(t)$ is finite, or there is nothing to prove; so $f^*(t) \le f^{**}(t) < \infty$. Let $E = \{x \in X : |f(x)| > f^*(t)\}$ and $t_0 = \rho(E)$. By Proposition 4.1, $t_0 = \mu_f(f^*(t)) \le t$. As f and f^* are equimeasurable, and f^* is decreasing, it follows that $f^*(s) = f^*(t)$ for $t_0 < s \le t$.

As $|f\chi_E| \le |f|$, we see $(f\chi_E)^* \le f^*$. But, $f\chi_E$ is supported on a set of measure t_0. So, $(f\chi_E)^*(s) = 0$ for $s > t_0$. Thus,

$$\int_E |f|\,d\rho = \int_0^\infty (f\chi_E)^*(s)\,ds = \int_0^{t_0} (f\chi_E)^*(s)\,ds \le \int_0^{t_0} f^*(s)\,ds.$$

Define g and h by

$$g(x) = \max\big\{|f(x)| - f^*(t), 0\big\} \cdot \alpha(x),$$

$$h(x) = \min\big\{|f(x)|, f^*(t)\big\} \cdot \alpha(x),$$

where $\alpha(x)|f(x)| = f(x)$, so that $f = g + h$. Observe,

$$\|g\|_1 = \int_E |f|\, d\rho - \rho(E) f^*(t) \leq \int_0^{t_0} f^*(s)\, ds - t_0 f^*(t).$$

On the other hand, $\|h\|_\infty \leq f^*(t)$ is clear from construction. Therefore,

$$\alpha_t \leq \|g\|_1 + t\|h\|_\infty \leq \int_0^{t_0} f^*(s)\, ds + (t - t_0) f^*(t) = \int_0^t f^*(s)\, ds = t f^{**}(t).$$

□

Theorem 4.8. *The 2-star operator is sublinear, i.e., for any $f_1, f_2 : (X, \rho) \to \mathbb{C}$ and $t > 0$, $(f_1 + f_2)^{**}(t) \leq f_1^{**}(t) + f_2^{**}(t)$.*

Proof. Fix $t > 0$ and $\epsilon > 0$. By the preceding lemma, choose $g_1, g_2 \in L^1(X)$ and $h_1, h_2 \in L^\infty(X)$ so that $f_j = g_j + h_j$ and $\|g_j\|_1 + t\|h_j\|_\infty \leq t f_j^{**}(t) + \epsilon$ for $j = 1, 2$. Then,

$$\begin{aligned} t(f_1 + f_2)^{**}(t) &\leq \|g_1 + g_2\|_1 + t\|h_1 + h_2\|_\infty \\ &\leq \Big(\|g_1\|_1 + t\|h_1\|_\infty\Big) + \Big(\|g_2\|_1 + t\|h_2\|_\infty\Big) \\ &\leq t f_1^{**}(t) + t f_2^{**}(t) + 2\epsilon. \end{aligned}$$

As ϵ is arbitrary, this completes the proof. □

4.4 A Characterization of $L \log L$

The space Zygmund space $L \log L$ arises naturally in a number of ways, particularly interpolation results. However, the exact definition of the space differs with the given application, and most definitions are somewhat unwieldy. The definition we present here, and use for the remainder of the text, is less conceptually natural, but once certain properties are established, is much easier to use.

For this section, we restrict (X, ρ) to be a probability space. For functions f on X, $f^*(t) = 0$ for $t > 1$. So, for simplicity, we can think of f^* and f^{**} as functions defined only on $[0, 1]$.

Definition. For functions $f : (X, \rho) \to \mathbb{C}$ define $\|f\|_{L\log L}$ by

$$\|f\|_{L\log L} = \int_0^1 f^{**}(t)\, dt.$$

Define the Zygumnd space $L\log L(X)$ as the set of all functions $f : (X, \rho) \to \mathbb{C}$ with $\|f\|_{L\log L} < \infty$.

It is clear from what we know about the 2-star operator that $\|\cdot\|_{L\log L}$ is a norm and $L\log L(X)$ is a Banach space. Further, we know that if $|g| \leq |f|$ a.e.$[\rho]$ then $\|g\|_{L\log L} \leq \|f\|_{L\log L}$ and $|f_n| \uparrow |f|$ a.e.$[\rho]$ implies $\|f_n\|_{L\log L} \uparrow \|f\|_{L\log L}$. What is not clear is the reason for choosing this definition. This is explained by the following.

Theorem 4.9. *$f \in L\log L(X)$ if and only if*

$$\int_X |f(x)| \log^+ |f(x)|\, \rho(dx) < \infty,$$

where $\log^+(x) = \max(\log x, 0)$.

Proof. As the map $x \mapsto x\log^+ x$ is continuous, increasing, and has value 0 at $x = 0$, we have by Lemma 4.2 that $\int_X |f| \log^+ |f|\, d\rho$ is finite if and only if $\int_0^1 f^*(t) \log^+ f^*(t)\, dt$ is finite. On the other hand, changing the order of integration shows

$$\int_0^1 f^{**}(t)\, dt = \int_0^1 f^*(s) \int_s^1 \frac{1}{t}\, dt\, ds = \int_0^1 f^*(s) \log(1/s)\, ds.$$

Assume $\int_0^1 f^*(t) \log^+ f^*(t)\, dt$ is finite. Let $E = \{t \in (0, 1) : f^*(t) > t^{-1/2}\}$ and $F = (0, 1) - E$. Then,

$$\begin{aligned}
\int_0^1 f^*(t)\log(1/t)\,dt &\le \int_E f^*(t)\log(f^*(t)^2)\,dt + \int_F t^{-1/2}\log(1/t)\,dt \\
&\le 2\int_0^1 f^*(t)\log^+ f^*(t) + \int_0^1 t^{-1/2}\log(1/t)\,dt \\
&= 2\int_0^1 f^*(t)\log^+ f^*(t) + 4 < \infty.
\end{aligned}$$

Now suppose $\int_0^1 f^{**}(t)\,dt$ is finite. Then, $\|f\|_1 = \int_0^1 f^*(t)\,dt \le \int_0^1 f^{**}(t)\,dt < \infty$. If $\|f\|_1 = 0$ there is nothing to prove, so assume otherwise. Let $g = f/\|f\|_1$ so that $\|g\|_1 = 1$. Then, $g^*(t) \le g^{**}(t) \le \|g\|_1/t = 1/t$. Also,

$$\begin{aligned}
\int_0^1 g^*(t)\log^+ g^*(t)\,dt \le \int_0^1 g^*(t)\log^+(1/t)\,dt &= \int_0^1 g^*(t)\log(1/t)\,dt \\
&= \frac{1}{\|f\|_1}\int_0^1 f^*(t)\log(1/t)\,dt < \infty
\end{aligned}$$

But,

$$\begin{aligned}
\int_0^1 f^*(t)\log^+ f^*(t)\,dt &= \|f\|_1 \int_0^1 g^*(t)\log^+(\|f\|_1 g^*(t))\,dt \\
&\le \|f\|_1\Big[\int_0^1 g^*(t)\log^+\|f\|_1\,dt + \int_0^1 g^*(t)\log^+ g^*(t)\,dt\Big] \\
&= \|f\|_1\Big[\log^+\|f\|_1 + \int_0^1 g^*(t)\log^+ g^*(t)\,dt\Big] < \infty.
\end{aligned}$$

$\square$

The quantity $\int_X |f|\log^+|f|\,d\rho$ is often taken as the definition of $\|\cdot\|_{L\log L}$. Indeed, this quantity naturally arises in many arguments. However, it is clearly not a norm, and makes any deep analysis difficult.

Our next goal is to show how $L\log L$ is related to L^p. First, we prove a special case of Hardy's inequality [11].

Lemma 4.10. *Let $1 < p < \infty$ and ψ be a nonnegative, measurable function on $(0,1)$. Then,*

$$\left[\int_0^1 \Big(\frac{1}{t}\int_0^t \psi(s)\,ds\Big)^p dt\right]^{1/p} \lesssim \Big(\int_0^1 \psi(s)^p\,ds\Big)^{1/p},$$

where the underlying constants depend only on p.

Proof. Fix p. Let p' be the conjugate exponent of p; that is, $\frac{1}{p}+\frac{1}{p'}=1$. Write $\psi(s)=[s^{-1/pp'}][s^{1/pp'}\psi(s)]$ and apply Hölder to see

$$\begin{aligned}\frac{1}{t}\int_0^t \psi(s)\,ds &\le \Big(\frac{1}{t}\int_0^t s^{-1/p}\,ds\Big)^{1/p'}\Big(\frac{1}{t}\int_0^t s^{1/p'}\psi(s)^p\,ds\Big)^{1/p} \\ &= p'^{1/p'}t^{-1/p-1/pp'}\Big(\int_0^t s^{1/p'}\psi(s)^p\,ds\Big)^{1/p}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^1 \Big(\frac{1}{t}\int_0^t \psi(s)\,ds\Big)^p dt &\le p'^{p/p'}\int_0^1 t^{-1-1/p'}\int_0^t s^{1/p'}\psi(s)^p\,ds\,dt \\ &= p'^{p/p'}\int_0^1 s^{1/p'}\psi(s)^p\int_s^1 t^{-1-1/p'}\,dt\,ds \\ &= p'^{p/p'}\int_0^1 s^{1/p'}\psi(s)^p\big[p'(s^{-1/p'}-1)\big]\,ds \\ &\le p'^{p/p'}\int_0^1 s^{1/p'}\psi(s)^p\big[p's^{-1/p'}\big]\,ds \\ &= p'^p\int_0^1 \psi(s)^p\,ds.\end{aligned}$$

□

Theorem 4.11. *For any $1<p\le\infty$, $L^p(X)\subseteq L\log L(X)\subseteq L^1(X)$, with $\|f\|_1\le\|f\|_{L\log L}\lesssim\|f\|_p$.*

Proof. Fix $f:\mathbb{T}\to\mathbb{C}$. We have trivially that $\|f\|_1=\int_0^1 f^*(t)\,dt\le\int_0^1 f^{**}(t)\,dt=\|f\|_{L\log L}$.

Now let $1<p<\infty$. First, as $(0,1)$ is a probability space, we have by Hölder that $\|f\|_{L\log L}\le(\int_0^1 f^{**}(t)^p\,dt)^{1/p}$. Now apply Hardy's inequality with $\psi(t)=f^*(t)$ to see $(\int_0^1 f^{**}(t)^p\,dt)^{1/p}\lesssim(\int_0^1 f^*(t)^p\,dt)^{1/p}=\|f\|_p$. □

The principal reason for defining $L\log L$ as we have is the ease in which we gain interpolation results.

Theorem 4.12. *Let T be a sublinear operator which maps $L^1(X) \to L^{1,\infty}(X)$ and $L^p(X) \to L^{q,\infty}(X)$, for some $1 < p, q < \infty$. Then, $T : L\log L(X) \to L^1(X)$.*

Proof. Set $m = (\frac{1}{q} - 1)(\frac{1}{p} - 1)^{-1}$, which is positive and finite. By Lemma 4.5,

$$(Tf)^*(t) \lesssim \left[\frac{1}{t}\int_0^{t^m} f^*(s)\,ds + t^{-1/q}\int_{t^m}^1 s^{1/p} f^*(s)\,\frac{ds}{s}\right],$$

for all $0 < t < 1$. Note, the second integral's upper limit is now 1, instead of ∞, as f^* is supported on $[0,1]$. A simple change of variables gives

$$\begin{aligned}\int_0^1 \frac{1}{t}\int_0^{t^m} f^*(s)\,ds\,dt &= \frac{1}{m}\int_0^1 \frac{1}{u}\int_0^u f^*(s)\,ds\,du \\ &= \frac{1}{m}\int_0^1 f^{**}(u)\,du = \frac{1}{m}\|f\|_{L\log L}.\end{aligned}$$

On the other hand, using Fubini,

$$\begin{aligned}\int_0^1 t^{-1/q}\int_{t^m}^1 s^{1/p-1} f^*(s)\,ds\,dt &= \int_0^1 s^{1/p-1} f^*(s)\int_0^{s^{1/m}} t^{-1/q}\,dt\,ds \\ &= \frac{1}{1-1/q}\int_0^1 s^{1/p-1}s^{1/m-1/mq} f^*(s)\,ds \\ &= \frac{1}{1-1/q}\int_0^1 f^*(s)\,ds \\ &\le \frac{1}{1-1/q}\int_0^1 f^{**}(s)\,ds = \frac{1}{1-1/q}\|f\|_{L\log L}.\end{aligned}$$

Hence,

$$\|Tf\|_1 = \int_0^1 (Tf)^*(t)\,dt \lesssim \left(\frac{1}{m} + \frac{1}{1-1/q}\right)\|f\|_{L\log L}.$$

$\square$

Corollary 4.13. *Let T be a sublinear operator. If for some $1 < p, r < \infty$*

$$\left\| \left(\sum_{k=1}^{\infty} |Tf_k|^r \right)^{1/r} \right\|_{1,\infty} \lesssim \left\| \left(\sum_{k=1}^{\infty} |f_k|^r \right)^{1/r} \right\|_1 \quad \textit{and}$$
$$\left\| \left(\sum_{k=1}^{\infty} |Tf_k|^r \right)^{1/r} \right\|_p \lesssim \left\| \left(\sum_{k=1}^{\infty} |f_k|^r \right)^{1/r} \right\|_p,$$

then

$$\left\| \left(\sum_{k=1}^{\infty} |Tf_k|^r \right)^{1/r} \right\|_1 \lesssim \left\| \left(\sum_{k=1}^{\infty} |f_k|^r \right)^{1/r} \right\|_{L \log L}.$$

Proof. Recall Banach-valued functions $f \in \mathcal{M}(X, B)$ as in Theorem 1.11. Although we did not do so for stylistic purposes, this chapter could have been presented in this more general setting. For instance, for $f \in \mathcal{M}(X, B)$, define $\mu_f(\lambda) = \rho\{x \in X : \|f(x)\|_B > \lambda\}$ and $f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}$. In this manner, we could redo this entire chapter replacing $\mathbb{C}$ and $|\cdot|$ with B and $\|\cdot\|_B$, and everything would follow as before.

Specifically, the previous theorem holds; if T is sublinear operator mapping $L^1_B(X)$ to $L^{1,\infty}_B(X)$ and $L^p_B(X)$ to $L^{q,\infty}_B(X)$, then $T : L \log L_B(X) \to L^1_B(X)$. But, simply by definition, $f^*(t) = (\|f\|_B)^*(t)$, where $(\|f\|_B)^*$ is understood as the decreasing rearrangement of the map $x \mapsto \|f(x)\|_B$. Thus, $\|f\|_{L \log L_B} = \big\| \|f\|_B \big\|_{L \log L}$.

Let $B = \ell^r$. For $f \in \mathcal{M}(X, B)$, let $\overline{T}(f) = (Tf_1, Tf_2, \ldots)$, which is sublinear, because T is. By hypothesis, $\overline{T} : L^1_B(X) \to L^{1,\infty}_B(X)$ and $L^p_B(X) \to L^p_B(X)$. Thus, $\overline{T} : L \log L_B(X) \to L^1_B(X)$, which is what we wanted to prove. $\square$

4.5 The n-Star Operator and $L(\log L)^n$

To extend the definition of $L \log L$, we first must extend the definition of the 2-star operator. We remain with the convention that (X, ρ) is a probability space.

Definition. For $f : (X, \rho) \to \mathbb{C}$, let $f^{(*,1)}(t) = f^*(t)$ and for integers $n \geq 2$, set $f^{(*,n)}(t) = \frac{1}{t}\int_0^t f^{(*,n-1)}(s)\,ds$.

Proposition 4.14. *For any $f, f_k, g : (X, \rho) \to \mathbb{C}$ and $\alpha \in \mathbb{C}$,*

1. *$f^{(*,n)}$ is nonnegative, decreasing, and identically 0 if and only if $f = 0$ a.e.[ρ],*
2. *$f^{(*,n)} \leq f^{(*,n+1)}$,*
3. *$|f| \leq |g|$ a.e.[ρ] implies $f^{(*,n)} \leq g^{(*,n)}$ pointwise,*
4. *$(\alpha f)^{(*,n)} = |\alpha| f^{(*,n)}$,*
5. *$|f_k| \uparrow |f|$ a.e.[ρ] implies $f_k^{(*,n)} \uparrow f^{(*,n)}$ pointwise,*
6. *$(f+g)^{(*,n)} \leq f^{(*,n)} + g^{(*,n)}$ ($n \geq 2$ only).*

Proof. It is known that $f^{(*,1)} = f^*$ is decreasing. Assume $f^{(*,n-1)}$ is decreasing. Let $0 < t_1 < t_2$. Then, $f^{(*,n-1)}(s) \leq f^{(*,n-1)}(st_1/t_2)$ for any $s > 0$. Thus,

$$\begin{aligned} f^{(*,n)}(t_2) &= \frac{1}{t_2}\int_0^{t_2} f^{(*,n-1)}(s)\,ds \leq \frac{1}{t_2}\int_0^{t_2} f^{(*,n-1)}(st_1/t_2)\,ds \\ &= \frac{1}{t_1}\int_0^{t_1} f^{(*,n-1)}(u)\,du = f^{(*,n)}(t_1). \end{aligned}$$

By induction, $f^{(*,n)}$ is decreasing. This gives

$$f^{(*,n+1)}(t) = \frac{1}{t}\int_0^t f^{(*,n)}(s)\,ds \geq f^{(*,n)}(t)\frac{1}{t}\int_0^t ds = f^{(*,n)}(t).$$

All other properties are easily established by induction and that each is known to hold for $n = 1$ (or $n = 2$ in the case of (6)). □

Definition. For functions $f : (X, \rho) \to \mathbb{C}$ and integers $n \geq 0$, define $\|f\|_{L(\log L)^n}$ by

$$\|f\|_{L(\log L)^n} = \int_0^1 f^{(*,n+1)}(t)\,dt.$$

Define the Zygumnd space $L(\log L)^n(X)$ as the set of all functions f with $\|f\|_{L(\log L)^n} < \infty$.

We note that $L(\log L)^0(X) = L^1(X)$, which is a useful notational shortcut. As before, it is clear that $L(\log L)^n(X)$ is a Banach space, and $\|\cdot\|_{L(\log L)^n}$ is a norm with the additional properties that $|f| \leq |g|$ a.e.$[\rho]$ implies $\|f\|_{L(\log L)^n} \leq \|g\|_{L(\log L)^n}$ and $|f_k| \uparrow |f|$ a.e.$[\rho]$ implies $\|f_k\|_{L(\log L)^n} \uparrow \|f\|_{L(\log L)^n}$. Further, this definition is related to the intuitive value, as before.

Theorem 4.15. *$f \in L(\log L)^n(X)$ if and only if*

$$\int_X |f(x)|(\log^+ |f(x)|)^n \,\rho(dx) < \infty.$$

Proof. The $n = 0$ case is trivial, and the $n = 1$ is already known. So, fix $n \geq 2$. As the map $x \mapsto x(\log^+ x)^n$ is continuous, increasing, and has value 0 at $x = 0$, we have by Lemma 4.2 that $\int_X |f|(\log^+ |f|)^n \,d\rho$ is finite if and only if $\int_0^1 f^*(t)(\log^+ f^*(t))^n \,dt$ is finite. On the other hand, changing the order of integration several times shows

$$\int_0^1 f^{(*,n+1)}(t)\,dt = \frac{1}{n!}\int_0^1 f^*(t)\log(1/t)^n\,dt$$

Suppose $\int_0^1 f^*(t)(\log^+ f^*(t))^n \,dt$ is finite. Let $E = \{t \in (0,1) : f^*(t) > t^{-1/2}\}$ and $F = (0,1) - E$. Then,

$$\begin{aligned}
\int_0^1 f^*(t)\log(1/t)^n\,dt &\leq \int_E f^*(t)\log(f^*(t)^2)^n\,dt + \int_F t^{-1/2}\log(1/t)^n\,dt \\
&\leq 2^n \int_0^1 f^*(t)\big(\log^+ f^*(t)\big)^n + \int_0^1 t^{-1/2}\log(1/t)^n\,dt \\
&= 2^n \int_0^1 f^*(t)\big(\log^+ f^*(t)\big)^n + 2^{n+1}n! < \infty.
\end{aligned}$$

Now suppose $\int_0^1 f^{(*,n+1)}(t)\,dt$ is finite. Then, we have $\|f\|_1 = \int_0^1 f^*(t)\,dt \leq \int_0^1 f^{(*,n+1)}(t)\,dt < \infty$. If $\|f\|_1 = 0$ there is nothing to prove, so assume otherwise. Let $g = f/\|f\|_1$ so that $\|g\|_1 = 1$. Then, $g^*(t) \leq g^{**}(t) \leq \|g\|_1/t = 1/t$. Also,

$$\int_0^1 g^*(t)\big(\log^+ g^*(t)\big)^n\,dt \le \int_0^1 g^*(t)\log^+(1/t)^n\,dt = \int_0^1 g^*(t)\log(1/t)^n\,dt$$
$$= \frac{1}{\|f\|_1}\int_0^1 f^*(t)\log(1/t)^n\,dt < \infty$$

But,

$$\begin{aligned}
&\int_0^1 f^*(t)\big(\log^+ f^*(t)\big)^n\,dt \\
&\qquad = \|f\|_1 \int_0^1 g^*(t)\big(\log^+(\|f\|_1 g^*(t))\big)^n\,dt \\
&\qquad \lesssim \|f\|_1\Big[\int_0^1 g^*(t)\big(\log^+\|f\|_1\big)^n\,dt + \int_0^1 g^*(t)\big(\log^+ g^*(t)\big)^n\,dt\Big] \\
&\qquad = \|f\|_1\Big[\big(\log^+\|f\|_1\big)^n + \int_0^1 g^*(t)\big(\log^+ g^*(t)\big)^n\,dt\Big] < \infty.
\end{aligned}$$

The transition from the second line to the third line follows from the fact that $(a+b)^r \le 2^{r-1}[a^r + b^r]$ for any $a, b \ge 0$ and $r \in \mathbb{R}$, which is proven by elementary calculus. $\square$

Theorem 4.16. *For any* $1 < p \le \infty$ *and* $n \ge 0$

$$L^p(X) \subseteq L(\log L)^{n+1}(X) \subseteq L(\log L)^n(X) \subseteq L^1(X),$$

with $\|f\|_1 \le \|f\|_{L(\log L)^n} \le \|f\|_{L(\log L)^{n+1}} \lesssim \|f\|_p$.

Proof. Fix $f : \mathbb{T} \to \mathbb{C}$ and $n \ge 0$. Note, $\|f\|_1 = \int_0^1 f^*(t)\,dt \le \int_0^1 f^{(*,n+1)}(t)\,dt = \|f\|_{L(\log L)^n}$. By the same token, $\|f\|_{L(\log L)^n} = \int_0^1 f^{(*,n+1)}(t)\,dt \le \int_0^1 f^{(*,n+2)}(t)\,dt = \|f\|_{L(\log L)^{n+1}}$.

Now let $1 < p < \infty$. First, as $(0,1)$ is a probability space, we have by Hölder that $\|f\|_{L(\log L)^n} \le (\int_0^1 f^{(*,n+1)}(t)^p\,dt)^{1/p} = \|f^{(*,n+1)}\|_p$. Applying Hardy's inequality (Lemma 4.10) with $\psi(t) = f^{(*,m)}(t)$ gives $\|f^{(*,m+1)}\|_p \lesssim \|f^{(*,m)}\|_p$. Iterating this we have $\|f\|_{L(\log L)^n} \le \|f^{(*,n+1)}\|_p \lesssim \|f^{(*,n)}\|_p \lesssim \cdots \lesssim \|f^{(*,1)}\|_p = \|f\|_p$. $\square$

An interpolation result can also be proven for $L(\log L)^n$. First, we need to find an estimate similar to the one before.

Lemma 4.17. *Let T be a sublinear operator which maps $L^1(X) \to L^{1,\infty}(X)$ and $L^p(X) \to L^{q,\infty}(X)$, for some $1 < p, q < \infty$. Then, for $n \geq 1$,*

$$(Tf)^{(*,n)}(t) \lesssim \left[\frac{1}{t}\int_0^{t^m} f^{(*,n)}(s)\,ds + t^{-1/q}\int_{t^m}^1 s^{1/p-1} f^{(*,n)}(s)\,ds\right], \quad 0 < t < 1,$$

where $m = (\frac{1}{q} - 1)(\frac{1}{p} - 1)^{-1}$.

Proof. The $n = 1$ case is precisely Lemma 4.5 (on a probability space) with $p_0 = q_0 = 1$. So, assume it is true for $n - 1$. Then,

$$\begin{aligned}
(Tf)^{(*,n)}(t) &= \frac{1}{t}\int_0^t T^{(*,n-1)}(s)\,ds \\
&\lesssim \frac{1}{t}\int_0^t \frac{1}{s}\int_0^{s^m} f^{(*,n-1)}(u)\,du\,ds + \frac{1}{t}\int_0^t s^{-1/q}\int_{s^m}^1 u^{1/p-1} f^{(*,n-1)}(u)\,du\,ds \\
&=: I + II.
\end{aligned}$$

By the change of variables $r = s^m$,

$$I = \frac{1}{m}\frac{1}{t}\int_0^{t^m} \frac{1}{r}\int_0^r f^{(*,n-1)}(u)\,du\,dr = \frac{1}{m}\frac{1}{t}\int_0^{t^m} f^{(*,n)}(r)\,dr.$$

On the other hand, changing the order of integration gives

$$\begin{aligned}
II &= \frac{1}{t}\int_0^{t^m} u^{1/p-1} f^{(*,n-1)}(u)\int_0^{u^{1/m}} s^{-1/q}\,ds\,du \\
&\quad + \frac{1}{t}\int_{t^m}^1 u^{1/p-1} f^{(*,n-1)}(u)\int_0^t s^{-1/q}\,ds\,du \\
&= \frac{1}{1-1/q}\,\frac{1}{t}\int_0^{t^m} f^{(*,n-1)}(u)\,du + \frac{1}{1-1/q}\,t^{-1/q}\int_{t^m}^1 u^{1/p-1} f^{(*,n-1)}(u)\,du \\
&\leq \frac{1}{1-1/q}\left[\frac{1}{t}\int_0^{t^m} f^{(*,n)}(u)\,du + t^{-1/q}\int_{t^m}^1 u^{1/p-1} f^{(*,n)}(u)\,du\right].
\end{aligned}$$

□

Theorem 4.18. *Let T be a sublinear operator which maps $L^1(X) \to L^{1,\infty}(X)$ and $L^p(X) \to L^{q,\infty}(X)$, for some $1 < p, q < \infty$. Then, for all $n \in \mathbb{N}$, we have $T : L(\log L)^n(X) \to L(\log L)^{n-1}(X)$.*

Proof. Set $m = (\frac{1}{q} - 1)(\frac{1}{p} - 1)^{-1}$, which is positive and finite. Using Lemma 4.17 and the same change of variables and Fubini arguments,

$$\begin{aligned}
\|Tf\|_{L(\log L)^{n-1}} &= \int_0^1 (Tf)^{(*,n)}(t)\,dt \\
&\lesssim \int_0^1 \frac{1}{t} \int_0^{t^m} f^{(*,n)}(s)\,ds\,dt + \int_0^1 t^{-1/q} \int_{t^m}^1 s^{1/p-1} f^{(*,n)}(s)\,ds\,dt \\
&= \frac{1}{m} \int_0^1 \frac{1}{u} \int_0^u f^{(*,n)}(s)\,ds\,du + \int_0^1 s^{1/p-1} f^{(*,n)}(s) \int_0^{s^{1/m}} t^{-1/q}\,dt\,ds \\
&= \frac{1}{m} \int_0^1 f^{(*,n+1)}(u)\,du + \frac{1}{1-1/q} \int_0^1 f^{(*,n)}(s)\,ds \lesssim \|f\|_{L(\log L)^n}.
\end{aligned}$$

□

Corollary 4.19. *Let T be a sublinear operator. If for some $1 < p, r < \infty$*

$$\Big\| \Big(\sum_{k=1}^\infty |Tf_k|^r \Big)^{1/r} \Big\|_{1,\infty} \lesssim \Big\| \Big(\sum_{k=1}^\infty |f_k|^r \Big)^{1/r} \Big\|_1 \quad \textit{and}$$
$$\Big\| \Big(\sum_{k=1}^\infty |Tf_k|^r \Big)^{1/r} \Big\|_p \lesssim \Big\| \Big(\sum_{k=1}^\infty |f_k|^r \Big)^{1/r} \Big\|_p,$$

then for all $n \in \mathbb{N}$

$$\Big\| \Big(\sum_{k=1}^\infty |Tf_k|^r \Big)^{1/r} \Big\|_{L(\log L)^{n-1}} \lesssim \Big\| \Big(\sum_{k=1}^\infty |f_k|^r \Big)^{1/r} \Big\|_{L(\log L)^n}.$$

4.6 $L \log L(\mathbb{T})$ and Connections to Hardy-Littlewood

Let us consider the probability space $(\mathbb{T}, m)$ and $L \log L(\mathbb{T})$. The maximal operator M maps $L^1 \to L^{1,\infty}$ and $L^p \to L^p$ for all $1 < p < \infty$. Therefore, by our interpolation results, $M : L \log L(\mathbb{T}) \to L^1$. However, much more can be said.

Theorem 4.20. *For any $0 < t < 1$, $f^{**}(t) \sim (Mf)^*(t)$, where the underlying constants do not depend on f or t.*

Proof. Fix t and f. We start by proving $(Mf)^*(t) \lesssim f^{**}(t)$. Let $\epsilon > 0$. By Lemma 4.7, there are functions g, h so that $f = g+h$ and $\|g\|_1 + t\|h\|_\infty \leq tf^{**}(t) + \epsilon$. On the other hand,

$$
\begin{aligned}
(Mf)^*(t) &\leq (Mg)^*(t/2) + (Mh)^*(t/2) = \frac{2}{t}\Big[\frac{t}{2}(Mg)^*(t/2)\Big] + (Mh)^*(t/2) \\
&\leq \frac{2}{t}\|Mg\|_{1,\infty} + \|Mh\|_\infty \lesssim \frac{2}{t}\|g\|_1 + \|h\|_\infty \\
&\leq \frac{2}{t}\big[\|g\|_1 + t\|h\|_\infty\big] \leq 2f^{**}(t) + 2\epsilon/t.
\end{aligned}
$$

Letting $\epsilon \to 0$, we have the first inequality.

For the second inequality, we may assume $(Mf)^*(t)$ is finite, or there is nothing to prove. Set Ω to be the closure of $\{Mf > (Mf)^*(t)\}$. Note that $|\Omega| = \mu_{Mf}((Mf)^*(t)) \leq t$. First, suppose $|\Omega| = 0$. Then, $|f| \leq Mf \leq (Mf)^*(t)$ a.e., which implies $f^*(s) \leq (Mf)^*(t)$ for all s. So, $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds \leq (Mf)^*(t)$.

Now, assume $|\Omega| > 0$. As $|\Omega| \leq t < 1$, for each $x \in \Omega$ we can choose an interval I_x which contains x in its interior and $I_x \cap \Omega^c \neq \emptyset$, but also so that most of I_x is in Ω. In particular, $|I_x| \leq 2|I_x \cap \Omega|$. Then, the interiors of $\{I_x : x \in \Omega\}$ cover Ω. As Ω is compact, we can choose a finite subcover $I_1, \dots, I_n$. Further, we can choose this subcover to be minimal, in that any point is contained in at most two of the I_k (this property is inherited from $\mathbb{R}$). On the other hand, as $I_j \cap \Omega^c \neq \emptyset$, there is a $y \in I_j \cap \Omega^c$. This implies $|I_j|^{-1}\int_{I_j} |f|\,dm \leq Mf(y) \leq (Mf)^*(t)$.

Define $g = f\chi_\Omega$ and $h = f\chi_{\Omega^c}$. We have immediately that $\|h\|_\infty = \|f\chi_{\Omega^c}\|_\infty \leq (Mf)^*(t)$. On the other hand,

$$\|g\|_1 \le \sum_{j=1}^{n} \int_{I_j} |f(x)|\,dx \le \sum_{j=1}^{n} (Mf)^*(t)|I_j|$$
$$\le 2(Mf)^*(t) \sum_{j=1}^{n} |I_j \cap \Omega| \le 4|\Omega|(Mf)^*(t) \le 4t(Mf)^*(t),$$

where the next to last inequality is gained from I_j being a minimal subcover. As $f = g + h$, it follows from Lemma 4.7 that $tf^{**}(t) \le \|g\|_1 + t\|h\|_\infty \lesssim t(Mf)^*(t)$. This completes the proof. □

Corollary 4.21. *For any $f : \mathbb{T} \to \mathbb{C}$, $f \in L\log L(\mathbb{T})$ if and only if $Mf \in L^1(\mathbb{T})$. In particular, $\|f\|_{L\log L} \sim \|Mf\|_1$.*

Proof. Using the previous theorem, $\|f\|_{L\log L} = \int_0^1 f^{**}(t)\,dt \sim \int_0^1 (Mf)^*(t)\,dt = \|Mf\|_1$. □

We note that $\|M(\cdot)\|_1$ is itself a norm, with the additional properties that $|f| \le |g|$ a.e. implies $\|Mf\|_1 \le \|Mg\|_1$ and $|f_n| \uparrow |f|$ a.e. implies $\|Mf_n\|_1 \uparrow \|Mf\|_1$. Therefore, on $\mathbb{T}$, we could have defined $\|f\|_{L\log L} = \|Mf\|_1$ and $L\log L(\mathbb{T})$ the space of functions which are mapped into L^1 by M. There is a similar result for $L(\log L)^n(\mathbb{T})$.

Corollary 4.22. *$f \in L(\log L)^{n+1}(\mathbb{T})$ if and only if $Mf \in L(\log L)^n(\mathbb{T})$ and $\|f\|_{L(\log L)^{n+1}} \sim \|Mf\|_{L(\log L)^n}$.*

Proof. We know $(Mf)^{(*,1)} \sim f^{(*,2)}$. It follows by induction that $(Mf)^{(*,n)} \sim f^{(*,n+1)}$ for all $n \ge 1$. Thus, $\|f\|_{L(\log L)^n} = \int_0^1 f^{(*,n+1)}(t)\,dt \sim \int_0^1 (Mf)^{(*,n)}(t)\,dt = \|Mf\|_{L(\log L)^{n-1}}$. □

Finally, we return to the unanswered question of the end-point estimates of the strong maximal function M_S. The probability space we focus on now is $(\mathbb{T}^d, m)$. As each of the j^{th} parameter maximal operators M_j map L^1 to weak-L^1 and L^p to

L^p, we have by interpolation that $M_j : L(\log L)^{n+1}(\mathbb{T}^d) \to L(\log L)^n(\mathbb{T}^d)$. Thus, for $n \geq d$,

$$\begin{aligned}
\|M_S\|_{L(\log L)^n} &\leq \|M_1 \circ M_2 \circ \cdots \circ M_d f\|_{L(\log L)^n} \\
&\lesssim \|M_2 \circ \cdots \circ M_d f\|_{L(\log L)^{n-1}} \\
&\lesssim \cdots \lesssim \|f\|_{L(\log L)^{n-d}}.
\end{aligned}$$

In particular, $M_S : L(\log L)^d(\mathbb{T}^d) \to L^1(\mathbb{T}^d)$ and $M_S : L(\log L)^{d-1}(\mathbb{T}^d) \to L^{1,\infty}(\mathbb{T}^d)$.

Chapter 5
Single-parameter Multipliers

5.1 Shifted Max and Square Operators

For $n \in \mathbb{Z}$, define the n-shifted maximal operator as

$$M^n f(x) = \sup_{x \in I} \frac{1}{|I|} \int_{I^n} |f(x)|\, dx,$$

where the supremum is taken over all intervals I containing x, but the integral is over I^n. We would like to establish results for M^n similar to those of M. This is quite simple.

Fix f and n. Let $x \in \mathbb{T}$ and $\epsilon > 0$. Choose an interval I containing x so that $M^n f(x) \leq |I|^{-1} \int_{I^n} |f(x)|\, dx + \epsilon$. There exists an interval I' (possibly all of $\mathbb{T}$) which contains both I and I^n, and $|I'| \leq (|n|+1)|I|$. Thus,

$$M^n f(x) - \epsilon \leq \frac{1}{|I|} \int_{I^n} |f(x)|\, dx \leq (|n|+1) \frac{1}{|I'|} \int_{I'} |f(x)|\, dx \leq (|n|+1) M f(x).$$

As ϵ is arbitrary, we have the pointwise estimate $M^n f \leq (|n|+1) M f$. Therefore, we immediately obtain all the L^p estimates of M, along with the Fefferman-Stein inequalities, for M^n with an additional factor of $|n|+1$.

Now consider an adapted family φ_I. By precisely the same argument used in Proposition 1.8,

$$M'^n f := \sup_I \frac{1}{|I|} \langle \varphi_{I^n}, f \rangle \chi_I \lesssim M^n f.$$

So, $M'^n f$ is also easily understood.

However, the shifted square function

$$S^n f(x) = \left(\sum_I \frac{|\langle \phi_{I^n}, f \rangle|^2}{|I|} \chi_I(x) \right)^{1/2}$$

does not permit a simple pointwise estimate. To prove the desired L^p results, one has to go through the argument as presented in Chapter 3 with S^n instead of S. We refrain from doing this, as only a brief description seems necessary.

It can be shown that $S^n : L^2 \to L^2$ exactly as before, with no dependence on n. This is because in the proof of Theorem 3.4 (and the preceding lemmas), we sum over all I with the same lengths, and the shift will not be important.

Fix a dyadic interval I and a an L^1-function supported on I with integral 0. Define $I^* = (2|n| + 2)I$ if $2|n| + 2 \leq 1/|I|$ and $I^* = \mathbb{T}$ otherwise. Then, $|I^*| \leq (2|n| + 2)|I|$. If J is a dyadic interval with $|J| < |I|$, we have that $J \subset I^*$ or J, I^* are disjoint. If it is the later, then by construction, J^n and $2I$ are disjoint. It now follows by precisely the same argument as in the proof of Lemma 3.6 that $\|S^n a\|_{L^1(\mathbb{T} - I^*)} \lesssim \|a\|_1$, where the underlying constant is independent of n. Applying the same decomposition as Theorem 3.7, we have $\|S^n f\|_{1,\infty} \lesssim (|n| + 1)\|f\|_1$ for all $f \in L^1$.

Define the shifted linearization

$$T_\epsilon^n f(x) = \sum_I \epsilon_I \langle \phi^1_{I^n}, f \rangle \phi^2_I(x).$$

By the same technique as before, $T_\epsilon^n : L^2 \to L^2$ with no dependence on n. For the weak-L^1 result, simply replace S^1 with $S^{1,n}$ in the proof of Theorem 3.10. The constant C which is chosen at the beginning will now depend on n, but as we saw, C actually cancels out by the end. This gives $\|T_\epsilon^n f\|_{1,\infty} \lesssim (|n| + 1)\|f\|_1$. The rest of the arguments follow as before giving $\|S^n f\|_p \lesssim (|n| + 1)\|f\|_p$ and $\|T_\epsilon^n f\|_p \lesssim (|n| + 1)\|f\|_p$. The Fefferman-Stein inequalities also hold for S^n, with the additional factor of $|n| + 1$.

On a different note, let $\alpha \in [0,1]$ and $I_\alpha = I + \alpha|I|$. This shifts the interval, much like I^n, but we use a different notation to distinguish the roles α and n will play. Define

$$M^n_\alpha f(x) = \sup_{x\in I} \frac{1}{|I|}\int_{I^n_\alpha} |f(y)|\,dy.$$

By the same argument as before, $M^n_\alpha f \leq (|n|+\alpha+1)Mf(x) \lesssim (|n|+1)Mf(x)$. So, if we let $M^{[n]}f(x) = \sup_\alpha M^n_\alpha f(x)$ for each x, then $M^{[n]}$ satisfies all the estimates of M ($L^p \to L^p$, $L^1 \to L^{1,\infty}$, and Fefferman-Stein inequalities) with an additional factor of $|n|+1$.

For an adapted family $\{\varphi_I\}$, let $\varphi_{I_\alpha}(x) = \varphi_I(x - \alpha|I|)$ so that each φ_{I_α} is uniformly adapted to I_α. Like the argument before, $M'^n_\alpha f(x) = \sup_I \frac{1}{|I|}\langle \varphi_{I^n_\alpha}, f\rangle \chi_I(x) \lesssim M^n_\alpha f(x)$. For a 0-mean family, let

$$S^n_\alpha f(x) = \Big(\sum_I \frac{|\langle \phi_{I^n_\alpha}, f\rangle|^2}{|I|}\chi_I(x)\Big)^{1/2}$$

and $S^{[n]}f(x) = \sup_\alpha S^n_\alpha f(x)$. We are interested in gaining estimates on $S^{[n]}$. First, fix an interval I. Note, for any x, $\operatorname{dist}(x, I_\alpha) \geq \operatorname{dist}(x,I) - \alpha|I|$ and

$$\begin{aligned}|\varphi_{I_\alpha}(x)| &\leq C_m\Big(1 + \frac{\operatorname{dist}(x,I_\alpha)}{|I|}\Big)^{-m} \leq 2^m C_m\Big(2 + \frac{\operatorname{dist}(x,I_\alpha)}{|I|}\Big)^{-m}\\ &\leq 2^m C_m\Big(2 - \alpha + \frac{\operatorname{dist}(x,I)}{|I|}\Big)^{-m} \leq 2^m C_m\Big(1 + \frac{\operatorname{dist}(x,I)}{|I|}\Big)^{-m}.\end{aligned}$$

That is, each φ_{I_α} is actually uniformly adapted to I. Fix f and n. For each dyadic interval I, choose an $I_\#$, dependent on f, so that $|\langle \phi_{I^n_\#}, f\rangle| = \sup_\alpha |\langle \phi_{I^n_\alpha}, f\rangle|$. Then,

$$S^{[n]}f(x) \leq \Big(\sum_I \frac{|\langle \phi_{I^n_\#}, f\rangle|^2}{|I|}\chi_I(x)\Big)^{1/2}.$$

As each $\varphi_{I^n_\#}$ is uniformly adapted to I^n, we observe that $S^{[n]}f$ is bounded by a kind of $S^n f$, with a new adapted family. Hence, $\|S^{[n]}f\|_{1,\infty} \lesssim (|n|+1)\|f\|_1$ and

$\|S^{[n]}f\|_p \lesssim (|n|+1)\|f\|_p$ as before.

Finally, let

$$T_\epsilon^{[n]} f(x) = \int_0^1 \sum_I \epsilon_I \langle \phi^1_{I^n_\alpha}, f\rangle \phi^2_{I_\alpha}(x)\, d\alpha.$$

Let $1 < p < \infty$ and take $\|g\|_{p'} \leq 1$. Then, by the normal Hölder argument $|\langle T_\epsilon^{[n]} f, g\rangle| \leq \|S^{[n]}f\|_p \|S^{[0]}g\|_{p'} \lesssim (|n|+1)\|f\|_p$. As g in the unit ball of $L^{p'}$ is arbitrary, $\|T_\epsilon^{[n]}f\|_p \lesssim (|n|+1)\|f\|_p$. To show that $\|T_\epsilon^{[n]}f\|_{1,\infty} \lesssim (|n|+1)\|f\|_1$, one needs to run the argument of Theorem 3.10 again, this time with S^1 replaced by $S^{1,[n]}$ and $S^{2,k}$ replaced by $S^{2,k,[0]}$. As each of the square functions is the supremum over α, the integral over α will be irrelevant.

5.2 Marcinkiewicz Multipliers

Definition. Let $m : \mathbb{R} \to \mathbb{C}$ be smooth away from 0 and uniformly bounded. We say m is a Marcinkiewicz multiplier if $|m^{(l)}(t)| \lesssim |t|^{-l}$ for $0 \leq l \leq 4$.

The restriction $l \leq 4$ is what we will need. It can often be assumed to hold for many more derivatives. Our definition here differs slightly from the classical definition. Normally, m is taken only to be in L^∞, not uniformly bounded. Typically, the multiplier appears in some integral and the value of m at 0 is irrelevant. Here, however, it will applied in a sum and the value is important.

Given a Marcinkiewicz multiplier m, define the Marcinkiewicz multiplier operator for $f \in L^1(\mathbb{T})$ as

$$\Lambda_m f(x) = \sum_{t\in\mathbb{Z}} m(t)\widehat{f}(t)e^{2\pi i t x}.$$

We will show this operator satisfies the same L^p properties as its classical counterpart on $\mathbb{R}$. First, we show the following technical results.

Lemma 5.1. *Fix positive integers k and K. For each $\vec{n} \in \mathbb{Z}^K$, write $\alpha(\vec{n}) = \prod_{j=1}^{K}(|n_j|+1)$. Suppose we have $f_{\vec{n}} : \mathbb{T}^d \to \mathbb{C}$ for each $\vec{n} \in \mathbb{Z}^K$ and $\|f_{\vec{n}}\|_{p,\infty} \leq \alpha(\vec{n})$ for all $\vec{n}$ and some $p \geq 1/k$. Set $r = k+3$ and $F = \sum_{\vec{n}} \alpha(\vec{n})^{-r} f_{\vec{n}}$. Then, $\|F\|_{p,\infty} \lesssim 1$.*

Proof. Let $\lambda > 0$. Fix $C = \sum_{\vec{n}} \alpha(\vec{n})^{-3/2}$. It is clear that

$$\{|F| > \lambda\} \subseteq \bigcup_{\vec{n}} \{|f_{\vec{n}}| > \lambda C^{-1} \alpha(\vec{n})^{r-3/2}\}.$$

So, $|\{|F| > \lambda\}| \leq \sum_{\vec{n}} |\{|f_{\vec{n}}| > \lambda C^{-1}\alpha(\vec{n})^{r-3/2}\}| \leq \frac{C^p}{\lambda^p} \sum_{\vec{n}} \|f_{\vec{n}}\|_{p,\infty}^p \alpha(\vec{n})^{-rp+3p/2} \leq \frac{C^p}{\lambda^p} \sum_{\vec{n}} \alpha(\vec{n})^{-rp+5p/2} \lesssim \lambda^{-p}$, because $p(-r+5/2) = p(-k-1/2) < -1$. As λ is arbitrary, $\|F\|_{p,\infty} \lesssim 1$. $\square$

Lemma 5.2. *Let m be any Marcinkiewicz multiplier and ψ_k^1 the functions guaranteed by Theorem 1.4. For each $k \in \mathbb{N}$, there is a smooth function m_k so that $m_k\widehat{\psi_k^1} = m\widehat{\psi_k^1}$ and*

$$m_k(t) = \sum_{n \in \mathbb{Z}} c_{k,n} e^{-2\pi i n 2^{-k} t},$$

where $|c_{k,n}| \lesssim (|n|+1)^{-4}$ uniformly in k.

Proof. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be smooth, with $\mathrm{supp}(\varphi) \subseteq [-1/2, -1/32] \cup [1/32, 1/2]$ and $\varphi = 1$ on $[-1/4, -1/16] \cup [1/16, 1/4]$. Define $m_k(t) = m(t)\varphi(2^{-k}t)$. Then, $m_k = m$ on $[-2^{k-2}, -2^{k-4}] \cup [2^{k-4}, 2^{k-2}]$, or equivalently $m_k\widehat{\psi_k^1} = m\widehat{\psi_k^1}$. Further, m_k is supported on $E_k := [-2^{k-1}, -2^{k-5}] \cup [2^{k-5}, 2^{k-1}] \subset [-2^{k-1}, 2^{k-1}]$, an interval of length 2^k.

Recall that $\{e^{-2\pi i n x}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for the Hilbert space $L^2([0,1])$, or any interval of length 1 in $\mathbb{R}$. Thus, $\{2^{-k/2} e^{-2\pi i n 2^{-k} x}\}$ is an orthonormal basis on any interval of length 2^k, and

$$m_k(t) = \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}} m_k(x) \frac{e^{2\pi i n 2^{-k} x}}{2^{k/2}}\, dx \right) \frac{e^{-2\pi i n 2^{-k} t}}{2^{k/2}} = \sum_{n\in\mathbb{Z}} c_{k,n} e^{-2\pi i n 2^{-k} t},$$

where $c_{k,n} = 2^{-k} \int_{\mathbb{R}} m_k(x) e^{2\pi i n 2^{-k} x}\, dx$.

First, if $n = 0$, then $c_{k,n} = 2^{-k} \int_{\mathbb{R}} m_k\, dm = 2^{-k} \int_{E_k} m_k\, dm$. So, $|c_{k,n}| \leq 2^{-k} |E_k| \|m\|_\infty \|\varphi\|_\infty \leq \|m\|_\infty \|\varphi\|_\infty \lesssim 1$.

Now assume $n \neq 0$. Let $C = \max\{\|\varphi^{(l)}\|_\infty : 0 \leq l \leq 4\}$. On E_k, $|m^{(l)}(x)| \lesssim |x|^{-l} \leq |2^{k-5}|^{-l} = 2^{-kl} 2^{5l}$ for $l \leq 4$. Thus,

$$|m_k^{(4)}(x)| \lesssim \sum_{l=0}^{4} |m^{(l)}(x)| |2^{-k(4-l)} \varphi^{(4-l)}(2^{-k} x)| \leq \sum_{l=0}^{4} 2^{-kl} 2^{5l} 2^{-4k} 2^{kl} C \lesssim 2^{-4k}.$$

By several iterations of integration by parts,

$$\begin{aligned} \left| \int_{\mathbb{R}} m_k(x) e^{2\pi i n 2^{-k} x}\, dx \right| &= \left| \int_{E_k} m_k(x) e^{2\pi i n 2^{-k} x}\, dx \right| \\ &= \left| \int_{E_k} m_k^{(4)}(x) \frac{e^{2\pi i n 2^{-k} x}}{(2\pi i n 2^{-k})^4}\, dx \right| \\ &\lesssim \frac{2^{4k}}{|n|^4} |E_k| \|m_k^{(4)}\|_\infty \lesssim \frac{2^k}{|n|^4} \lesssim \frac{2^k}{(|n|+1)^4}. \end{aligned}$$

Namely, $|c_{k,n}| \lesssim (|n|+1)^{-4}$. □

Theorem 5.3. *For any Marcinkiewicz multiplier m, $\Lambda_m : L^1(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})$ and $\Lambda_m : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ for $1 < p < \infty$.*

Proof. We start by noting that we can assume $m(0) = 0$. Let $m_0 = m$ away from 0 and $m_0(0) = 0$. Then, m_0 is a Marcinkiewicz multiplier and $\Lambda_m f(x) = m(0)\widehat{f}(0) + \Lambda_{m_0} f(x)$. But, $|m(0)\widehat{f}(0)| = |m(0) \int_{\mathbb{T}} f(x)\, dx| \lesssim \|f\|_1 \leq \|f\|_p$ for any p, as m is uniformly bounded. Thus, it suffices to prove the result for Λ_{m_0}, or equivalently, assuming $m(0) = 0$.

Fix $f, g \in L^1(\mathbb{T})$. Define the reflection of a function by $\widetilde{f}(x) = f(-x)$. Let $f_0 = \widetilde{\widetilde{f}}$ and $g_0 = \widetilde{\widetilde{g}}$. Then,

$$\begin{aligned}\langle \Lambda_m f, \widetilde{g} \rangle &= \int_{\mathbb{T}} \Lambda_m f(x) g_0(x)\, dx = \int_{\mathbb{T}} \left(\sum_{t\in\mathbb{Z}} m(t)\widehat{f}(t) e^{2\pi i x t} \right) g_0(x)\, dx \\ &= \sum_{t\in\mathbb{Z}} m(t)\widehat{f}(t) \int_{\mathbb{T}} g_0(x) e^{2\pi i x t}\, dx = \sum_{t\in\mathbb{Z}} m(t)\widehat{f}(t)\widehat{g_0}(-t).\end{aligned}$$

Now apply Theorem 1.4 to write

$$\begin{aligned}\langle \Lambda_m f, \widetilde{g} \rangle &= \sum_{t\in\mathbb{Z}} \sum_{k=1}^{\infty} m(t)\widehat{f}(t)\widehat{\psi_k^1}(t)\widehat{g_0}(-t)\widehat{\psi_k^2}(-t) \\ &= \sum_{k=1}^{\infty} \sum_{t\in\mathbb{Z}} m_k(t)\widehat{f}(t)\widehat{\psi_k^1}(t)\widehat{g_0}(-t)\widehat{\psi_k^2}(-t),\end{aligned}$$

where m_k is as given in Lemma 5.2. Let $\psi_{k,n}^1(x) = \psi_k^1(x - n2^{-k})$. Then,

$$\begin{aligned}\langle \Lambda_m f, \widetilde{g} \rangle &= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\infty} \sum_{t\in\mathbb{Z}} c_{k,n} e^{-2\pi i n 2^{-k} t} \widehat{f}(t)\widehat{\psi_k^1}(t)\widehat{g_0}(-t)\widehat{\psi_k^2}(-t) \\ &= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\infty} \sum_{t\in\mathbb{Z}} c_{k,n} \widehat{f}(t)\widehat{\psi_{k,n}^1}(t)\widehat{g_0}(-t)\widehat{\psi_k^2}(-t) \\ &= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\infty} \sum_{t\in\mathbb{Z}} c_{k,n} (f * \psi_{k,n}^1)\widehat{\ }(t)(g_0 * \psi_k^2)\widehat{\ }(-t) \\ &= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\infty} c_{k,n} \int_{\mathbb{T}} (f * \psi_{k,n}^1)(x)(g_0 * \psi_k^2)(x)\, dx,\end{aligned}$$

the last line being an application of Plancherel. Even though f, g_0 are only assumed in L^1, $f * \psi_{k,n}^1$ and $g_0 * \psi_k^2$ are smooth, thus in L^2. Focusing on just the integral portion,

$$
\begin{aligned}
\int_{\mathbb{T}} &(f * \psi^1_{k,n})(x)(g_0 * \psi^2_k)(x)\,dx \\
&= \int_0^1 (f * \psi^1_{k,n})(x)(g_0 * \psi^2_k)(x)\,dx \\
&= 2^{-k} \int_0^{2^k} (f * \psi^1_{k,n})(2^{-k}x)(g_0 * \psi^2_k)(2^{-k}x)\,dx \\
&= 2^{-k} \sum_{j=0}^{2^k-1} \int_j^{j+1} (f * \psi^1_{k,n})(2^{-k}x)(g_0 * \psi^2_k)(2^{-k}x)\,dx \\
&= 2^{-k} \sum_{j=0}^{2^k-1} \int_0^1 (f * \psi^1_{k,n})(2^{-k}(\alpha + j))(g_0 * \psi^2_k)(2^{-k}(\alpha + j))\,d\alpha \\
&= 2^{-k} \sum_{j=0}^{2^k-1} \int_0^1 \langle \psi^1_{k,j,n,\alpha}, \overline{f}\rangle \langle \psi^2_{k,j,\alpha}, \overline{g_0}\rangle\,d\alpha,
\end{aligned}
$$

where $\psi^1_{k,j,n,\alpha}(x) = \psi^1_{k,n}(2^{-k}(\alpha + j) - x) = \psi^1_k(2^{-k}(\alpha + j + n) - x)$ and $\psi^2_{k,j,\alpha}(x) = \psi^2_k(2^{-k}(\alpha + j) - x)$.

For a dyadic interval $I = [2^{-k}j, 2^{-k}(j + 1)]$, let $\varphi^2_{I_\alpha} = 2^{-k}\widetilde{\psi^2_{k,j,\alpha}}$. Similarly, let $\varphi^1_{I^n_\alpha} = 2^{-k}\widetilde{\psi^1_{k,j,n,\alpha}}$. It is easily checked that the original conditions on ψ^1, ψ^2 guarantee that φ^1_I, φ^2_I are 0-mean adapted families. Let $\phi^1_I = |I|^{-1/2}\varphi^1_I$ and $\phi^2_I = |I|^{-1/2}\varphi^2_I$, so that

$$
\begin{aligned}
\langle \Lambda_m f, \widetilde{g}\rangle &= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\infty} c_{k,n} 2^{-k} \sum_{j=0}^{2^k-1} \int_0^1 \langle \psi^1_{k,j,n,\alpha}, \overline{f}\rangle \langle \psi^2_{k,j,\alpha}, \overline{g_0}\rangle\,d\alpha \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 \sum_I c_{I,n} \langle \phi^1_{I^n_\alpha}, f_0\rangle \langle \phi^2_{I_\alpha}, g\rangle\,d\alpha,
\end{aligned}
$$

where the inner sum is over all dyadic intervals and $c_{I,n} = c_{k,n}$ when $|I| = 2^{-k}$. Write $c'_{I,n} = (|n|+1)^4 c_{I,n}$, which are uniformly bounded in I and n by Lemma 5.2. Hence,

$$
\begin{aligned}
\langle \Lambda_m f, \widetilde{g}\rangle &= \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^4}\int_0^1 \sum_I c'_{I,n}\langle \phi^1_{I^n_\alpha}, f_0\rangle\langle \phi^2_{I_\alpha}, g\rangle\, d\alpha \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^4}\langle T^{[n]}_{c'} f_0, g\rangle \\
&= \Big\langle \sum_{n\in\mathbb{Z}} \frac{1}{(|n|+1)^4} T^{[n]}_{c'} f_0, g\Big\rangle
\end{aligned}
$$

As $g \in L^1$ is arbitrary, it follows that $\widetilde{\Lambda_m f} = \sum(|n|+1)^{-4}T^{[n]}_{c'}f_0$ a.e.. But, $\|T^{[n]}_{c'}f_0\|_p \lesssim (|n|+1)\|f_0\|_p = (|n|+1)\|f\|_p$ and $\|T^{[n]}_{c'}f_0\|_{1,\infty} \lesssim (|n|+1)\|f\|_1$. So, we have immediately that $\|\Lambda_m f\|_p \lesssim \|f\|_p$ for all $1 < p < \infty$. Further, by Lemma 5.1 (with $K = k = 1$), $\|\Lambda_m f\|_{1,\infty} \lesssim \|f\|_1$. □

Corollary 5.4. $\Lambda_m : L(\log L)^n \to L(\log L)^{n-1}$ *for any Marcinkiewicz multiplier* m *and* $n \in \mathbb{N}$.

5.3 Single-parameter Paraproducts

Return to the linearization T_ϵ defined in Section 3.3. This linear operator can be viewed as the simplest in a family of multilinear operators, which we call paraproducts. For simplicity, we will focus only on the bilinear case, but the other operators are handled in precisely the same manner.

For $f, g : \mathbb{T} \to \mathbb{C}$, the single-parameter bilinear paraproducts are defined

$$
T^a_\epsilon(f,g)(x) = \sum_I \epsilon_I \frac{1}{|I|^{1/2}}\langle \phi^1_I, f\rangle\langle \phi^2_I, g\rangle \phi^3_I(x),
$$

for $a = 1, 2, 3$, where φ^1_I, φ^2_I, and φ^3_I are three adapted families with the property that $\int_{\mathbb{T}} \varphi^i_I\, dm = 0$ for $i \neq a$. As before, the sum is over all dyadic intervals I, and (ϵ_I) is a uniformly bounded sequence. By dividing out a constant, we can assume $|\epsilon_I| \leq 1$. The reason for the terminology single-parameter will be become clearer in the next chapter.

The primary goal of this section is to prove standard L^p estimates of these paraproducts, which we do now.

Theorem 5.5. *$T^a_\epsilon : L^{p_1} \times L^{p_2} \to L^p$ for $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. If p_1 or p_2 or both are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$. The underlying constants do not depend on a or the sequence ϵ_I.*

Proof. We will assume that $a = 1$, so that $\int \phi^i_I \, dm = 0$ for $i = 2, 3$. It will be clear that the proofs for $a = 2, 3$ are essentially the same.

First, suppose $p > 1$. Then, necessarily $p_1, p_2 > 1$ and $1 < p' < \infty$. Note, $1/p_1 + 1/p_2 + 1/p' = 1$. Fix $h \in L^{p'}(\mathbb{T})$ with $\|h\|_{p'} \leq 1$. Then,

$$
\begin{aligned}
|\langle T^1_\epsilon(f,g), h\rangle| &= \Big| \sum_I \epsilon_I \frac{1}{|I|^{1/2}} \langle \phi^1_I, f\rangle \langle \phi^2_I, g\rangle \langle \phi^3_I, h\rangle \Big| \\
&\leq \sum_I \frac{1}{|I|^{1/2}} |\langle \phi^1_I, f\rangle| |\langle \phi^2_I, g\rangle| |\langle \phi^3_I, h\rangle| \\
&= \int_{\mathbb{T}} \sum_I \frac{|\langle \phi^1_I, f\rangle|}{|I|^{1/2}} \frac{|\langle \phi^2_I, g\rangle|}{|I|^{1/2}} \frac{|\langle \phi^3_I, h\rangle|}{|I|^{1/2}} \chi_I(x) \, dx \\
&\leq \int_{\mathbb{T}} \left(\sup_I \frac{|\langle \phi^1_I, f\rangle|}{|I|^{1/2}} \chi_I(x) \right) \left(\sum_I \frac{|\langle \phi^2_I, g\rangle|^2}{|I|} \chi_I(x) \right)^{1/2} \left(\sum_I \frac{|\langle \phi^3_I, h\rangle|^2}{|I|} \chi_I(x) \right)^{1/2} dx \\
&= \int_{\mathbb{T}} M' f(x) S^2 g(x) S^3 h(x) \, dx \\
&\leq \|M' f\|_{p_1} \|S^2 g\|_{p_2} \|S^3 h\|_{p'} \lesssim \|f\|_{p_1} \|g\|_{p_2}.
\end{aligned}
$$

As h in the unit ball of $L^{p'}$ is arbitrary, we have $\|T^1_\epsilon(f,g)\|_p \lesssim \|f\|_{p_1} \|g\|_{p_2}$.

Now suppose $1/2 \leq p \leq 1$. We will show $T^1_\epsilon : L^{p_1} \times L^{p_2} \to L^{p,\infty}$ for all $1 \leq p_1, p_2 < \infty$. The fact that $L^{p,\infty}$ can be replaced by L^p where appropriate will follow immediately from interpolation of these results. Fix $1 \leq p_1, p_2 < \infty$.

Let $\|f\|_{p_1} = \|g\|_{p_2} = 1$ and $|E| > 0$. By Lemma 3.9, we will be done if we can find $E' \subseteq E$, $|E'| > |E|/2$ so that $|\langle T^1_\epsilon(f,g), \chi_{E'}\rangle| \lesssim 1 \leq |E|^{1-1/p}$. Using Theorem 1.10, decompose each ϕ^3_I into

$$\phi_I^3 = \sum_{k=1}^{\infty} 2^{-10k} \phi_I^{3,k}$$

where $\phi_I^{3,k}$ is the normalization of a 0-mean adapted family $\varphi_I^{3,k}$, which are uniformly adapted to I. Further, $\mathrm{supp}(\phi_I^{3,k}) \subseteq 2^k I$ for k small enough, while $\phi_I^{3,k}$ is identically 0 otherwise. Now write

$$\langle T_\epsilon^1(f,g), \chi_{E'} \rangle = \sum_{k=1}^{\infty} 2^{-10k} \sum_I \epsilon_I \frac{1}{|I|^{1/2}} \langle \phi_I^1, f \rangle \langle \phi_I^2, g \rangle \langle \phi_I^{3,k}, \chi_{E'} \rangle.$$

Hence, it suffices to show $|\sum \epsilon_I |I|^{-1/2} \langle \phi_I^1, f \rangle \langle \phi_I^2, g \rangle \langle \phi_I^{3,k}, \chi_{E'} \rangle| \lesssim 2^{4k}$, so long as the underlying constants are independent of k.

Let S^2 and $S^{3,k}$ be the square functions for ϕ_I^2 and $\phi_I^{3,k}$. For each $k \in \mathbb{N}$, define

$$\begin{aligned} \Omega_{-3k} &= \{Mf > C2^{3k}\} \cup \{S^2 g > C2^{3k}\}, \\ \widetilde{\Omega}_k &= \{M(\chi_{\Omega_{-3k}}) > 1/100\}, \\ \widetilde{\widetilde{\Omega}}_k &= \{M(\chi_{\widetilde{\Omega}_k}) > 2^{-k-1}\}. \end{aligned}$$

and

$$\Omega = \bigcup_{k \in \mathbb{N}} \widetilde{\widetilde{\Omega}}_k.$$

Observe, $|\Omega|$ is less than or equal to

$$100 \sum_{k=1}^{\infty} 2^{k+1} \|M\|_{L^1 \to L^{1,\infty}}^2 \left[\frac{1}{C^{p_1}} 2^{-3p_1 k} \|M\|_{L^{p_1} \to L^{p_1,\infty}}^{p_1} + \frac{1}{C^{p_2}} 2^{-3p_2 k} \|S^2\|_{L^{p_2} \to L^{p_2,\infty}}^{p_2} \right].$$

Therefore, we can choose C independent of f and g so that $|\Omega| < |E|/2$. Set $E' = E - \Omega = E \cap \Omega^c$. Then, $E' \subseteq E$ and $|E'| > |E|/2$.

Fix $k \in \mathbb{N}$. Set $Z_k = \{S^2 g = 0\} \cup \{S^{3,k}(\chi_{E'}) = 0\}$. Let $\mathcal{D}$ be any finite collection of dyadic intervals. We divide this collection into three subcollections. Set $\mathcal{D}_1 = \{I \in \mathcal{D} : I \cap Z_k \neq \emptyset\}$. For the remaining intervals, let $\mathcal{D}_2 = \{I \in \mathcal{D} - \mathcal{D}_1 : I \subseteq \widetilde{\Omega}_k\}$ and $\mathcal{D}_3 = \{I \in \mathcal{D} - \mathcal{D}_1 : I \cap \widetilde{\Omega}_k^c \neq \emptyset\}$.

If $I \in \mathcal{D}_1$, then there is some $x \in I \cap Z_k$, which implies $S^2 g(x) = 0$ or $S^{3,k}(\chi_{E'})(x) = 0$. If it is the first, $\langle \phi_J^2, g\rangle = 0$ for all dyadic J containing x. In particular, $\langle \phi_I^2, g\rangle = 0$. If it is the second, then $\langle \phi_I^{3,k}, \chi_{E'}\rangle = 0$. As this holds for all $I \in \mathcal{D}_1$, we have

$$\sum_{I \in \mathcal{D}_1} \frac{1}{|I|^{1/2}} |\langle \phi_I^1, f\rangle| |\langle \phi_I^2, g\rangle| |\langle \phi_I^{3,k}, \chi_{E'}\rangle| = 0.$$

Now suppose $I \in \mathcal{D}_2$, namely $I \subseteq \widetilde{\Omega}_k$. If k is big enough so that $2^k > 1/|I|$, then $\phi_I^{3,k}$ is identically 0 and $\langle \phi_I^{3,k}, \chi_{E'}\rangle = 0$. If $2^k \leq 1/|I|$, then $\phi_I^{3,k}$ is supported in $2^k I$. Let $x \in 2^k I$, and observe

$$M(\chi_{\widetilde{\Omega}_k})(x) \geq \frac{1}{|2^k I|} \int_{2^k I} \chi_{\widetilde{\Omega}_k} \, dm \geq \frac{1}{2^k} \frac{1}{|I|} \int_I \chi_{\widetilde{\Omega}_k} \, dm = 2^{-k} > 2^{-k-1}.$$

That is, $2^k I \subseteq \widetilde{\widetilde{\Omega}}_k \subseteq \Omega$, a set disjoint from E'. Thus, $\langle \phi_I^{3,k}, \chi_{E'}\rangle = 0$. As this holds for all $I \in \mathcal{D}_2$, we have

$$\sum_{I \in \mathcal{D}_2} \frac{1}{|I|^{1/2}} |\langle \phi_I^1, f\rangle| |\langle \phi_I^2, g\rangle| |\langle \phi_I^{3,k}, \chi_{E'}\rangle| = 0.$$

Finally, we concentrate on $\mathcal{D}_3$. Define Ω_{-3k+1} and Π_{-3k+1} by

$$\Omega_{-3k+1} = \{Mf > C 2^{3k-1}\},$$

$$\Pi_{-3k+1} = \{I \in \mathcal{D}_3 : |I \cap \Omega_{-3k+1}| > |I|/100\}.$$

Inductively, define for all $n > -3k+1$,

$$\Omega_n = \{Mf > C2^{-n}\},$$

$$\Pi_n = \{I \in \mathcal{D}_3 - \bigcup_{j=-3k+1}^{n-1} \Pi_j : |I \cap \Omega_n| > |I|/100\}.$$

As $\|f\|_{p_1} = 1$, and thus not equal to 0 a.e., $Mf > 0$ everywhere. So, it is clear that each $I \in \mathcal{D}_3$ will be in one of these collections.

Set $\Omega'_{-3k} = \Omega_{-3k}$ for symmetry. Define Ω'_{-3k+1} and Π'_{-3k+1} by

$$\Omega'_{-3k+1} = \{S^2 g > C2^{3k-1}\},$$

$$\Pi'_{-3k+1} = \{I \in \mathcal{D}_3 : |I \cap \Omega'_{-3k+1}| > |I|/100\}.$$

Inductively, define for all $n > -3k+1$,

$$\Omega'_n = \{S^2 g > C2^{-n}\},$$

$$\Pi'_n = \{I \in \mathcal{D}_3 - \bigcup_{j=-3k+1}^{n-1} \Pi'_j : |I \cap \Omega'_n| > |I|/100\}.$$

As every $I \in \mathcal{D}_3$ is not in $\mathcal{D}_1$, that is $S^2 g > 0$ on I, it is clear that each $I \in \mathcal{D}_3$ will be in one of these collections.

Now, we can choose an integer N big enough so that $\Omega''_{-N} = \{S^{3,k}(\chi_{E'}) > 2^N\}$ has very small measure. In particular, we take N big enough so that $|I \cap \Omega''_{-N}| < |I|/100$ for all $I \in \mathcal{D}_3$, which is possible since $\mathcal{D}_3$ is a finite collection. Define

$$\Omega''_{-N+1} = \{S^{3,k}(\chi_{E'}) > 2^{N-1}\},$$

$$\Pi''_{-N+1} = \{I \in \mathcal{D}_3 : |I \cap \Omega''_{-N+1}| > |I|/100\},$$

and

$$\Omega''_n = \{S^{3,k}(\chi_{E'}) > 2^{-n}\},$$

$$\Pi''_n = \{I \in \mathcal{D}_3 - \bigcup_{j=-N+1}^{n-1} \Pi''_j : |I \cap \Omega''_n| > |I|/100\},$$

Again, all $I \in \mathcal{D}_3$ must be in one of these collections.

Consider $I \in \mathcal{D}_3$, so that $I \cap \widetilde{\Omega}_k^c \neq \emptyset$. Then, there is some $x \in I \cap \widetilde{\Omega}_k^c$ which implies $|I \cap \Omega_{-3k}|/|I| \leq M(\chi_{\Omega_{-3k}})(x) \leq 1/100$. Write $\Pi_{n_1,n_2,n_3} = \Pi_{n_1} \cap \Pi'_{n_2} \cap \Pi''_{n_3}$. So,

$$\begin{aligned}
&\sum_{I \in \mathcal{D}_3} \frac{1}{|I|^{1/2}} |\langle \phi_I^1, f\rangle||\langle \phi_I^2, g\rangle||\langle \phi_I^{3,k}, \chi_{E'}\rangle| \\
&\qquad = \sum_{n_1,n_2 > -3k,\, n_3 > -N} \Bigg[\sum_{I \in \Pi_{n_1,n_2,n_3}} \frac{1}{|I|^{1/2}} |\langle \phi_I^1, f\rangle||\langle \phi_I^2, g\rangle||\langle \phi_I^{3,k}, \chi_{E'}\rangle| \Bigg] \\
&\qquad = \sum_{n_1,n_2 > -3k,\, n_3 > -N} \Bigg[\sum_{I \in \Pi_{n_1,n_2,n_3}} \frac{|\langle \phi_I^1, f\rangle|}{|I|^{1/2}} \frac{|\langle \phi_I^2, g\rangle|}{|I|^{1/2}} \frac{|\langle \phi_I^{3,k}, \chi_{E'}\rangle|}{|I|^{1/2}} |I| \Bigg].
\end{aligned}$$

Suppose $I \in \Pi_{n_1,n_2,n_3}$. If $n_1 > -3k+1$, then $I \in \Pi_{n_1}$, which in particular says $I \notin \Pi_{n_1-1}$. So, $|I \cap \Omega_{n_1-1}| \leq |I|/100$. If $n_1 = -3k+1$, then we still have $|I \cap \Omega_{-3k}| \leq |I|/100$, as $I \in \mathcal{D}_3$. Similarly, If $n_2 > -3k+1$, then $I \in \Pi'_{n_2}$, which in particular says $I \notin \Pi'_{n_2-1}$. So, $|I \cap \Omega'_{n_2-1}| \leq |I|/100$. If $n_2 = -3k+1$, then $|I \cap \Omega'_{-3k}| = |I \cap \Omega_{-3k}| \leq |I|/100$, as $I \in \mathcal{D}_3$. Finally, if $n_3 > -N+1$, then $I \notin \Pi''_{n_3-1}$ and $|I \cap \Omega''_{n_3-1}| \leq |I|/100$. If $n_3 = -N+1$, then $|I \cap \Omega''_{-N}| \leq |I|/100$ by the choice of N. So, $|I \cap \Omega^c_{n_1-1} \cap \Omega'^c_{n_2-1} \cap \Omega''^c_{n_3-1}| \geq \frac{97}{100}|I|$. Let $\Omega_{n_1,n_2,n_3} = \bigcup\{I : I \in \Pi_{n_1,n_2,n_3}\}$. Then,

$$|I \cap \Omega^c_{n_1-1} \cap \Omega'^c_{n_2-1} \cap \Omega''^c_{n_3-1} \cap \Omega_{n_1,n_2,n_3}| \geq \frac{97}{100}|I|$$

for all $I \in \Pi_{n_1,n_2,n_3}$. Further,

$$
\begin{aligned}
&\sum_{I\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_I^1,f\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^2,g\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^{3,k},\chi_{E'}\rangle|}{|I|^{1/2}}|I| \\
&\lesssim \sum_{I\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_I^1,f\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^2,g\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^{3,k},\chi_{E'}\rangle|}{|I|^{1/2}}|I\cap\Omega_{n_1-1}^c\cap\Omega_{n_2-1}'^c\cap\Omega_{n_3-1}''^c\cap\Omega_{n_1,n_2,n_3}| \\
&= \int_{\Omega_{n_1-1}^c\cap\Omega_{n_2-1}'^c\cap\Omega_{n_3-1}''^c\cap\Omega_{n_1,n_2,n_3}} \sum_{I\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_I^1,f\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^2,g\rangle|}{|I|^{1/2}}\frac{|\langle\phi_I^{3,k},\chi_{E'}\rangle|}{|I|^{1/2}}\chi_I(x)\,dx \\
&\lesssim \int_{\Omega_{n_1-1}^c\cap\Omega_{n_2-1}'^c\cap\Omega_{n_3-1}''^c\cap\Omega_{n_1,n_2,n_3}} Mf(x)S^2g(x)S^{3,k}(\chi_{E'})(x)\,dx \\
&\lesssim C^2 2^{-n_1}2^{-n_2}2^{-n_3}|\Omega_{n_1,n_2,n_3}|.
\end{aligned}
$$

Note, $|\Omega_{n_1,n_2,n_3}| \le |\bigcup\{I : I \in \Pi_{n_1}\}| \le |\{M(\chi_{\Omega_{n_1}}) > 1/100\}| \lesssim |\Omega_{n_1}| = |\{Mf > C2^{-n_1}\}| \lesssim C^{-p_1}2^{p_1n_1}$. By the same argument, $|\Omega_{n_1,n_2,n_3}| \lesssim |\Omega'_{n_2}| = |\{S^2g > C2^{-n_2}\}| \lesssim C^{-p_2}2^{p_2n_2}$, and $|\Omega_{n_1,n_2,n_3}| \lesssim |\Omega''_{n_3}| = |\{S^{3,k}(\chi_{E'}) > 2^{-n_3}\}| \lesssim 2^{\alpha n_3}$ for any $\alpha \ge 1$. Therefore, $|\Omega_{n_1,n_2,n_3}| \lesssim C^{-p_1-p_2}2^{\theta_1p_1n_1}2^{\theta_2p_2n_2}2^{\theta_3\alpha n_3}$ for any $\theta_1+\theta_2+\theta_3 = 1$, $0 \le \theta_1,\theta_2,\theta_3 \le 1$. Hence,

$$
\begin{aligned}
\sum_{I\in\mathcal{D}_3} &\frac{1}{|I|^{1/2}}|\langle\phi_I^1,f\rangle||\langle\phi_I^2,g\rangle||\langle\phi_I^{3,k},\chi_{E'}\rangle| \\
&\lesssim \sum_{n_1,n_2>-3k,\ n_3>0} 2^{(\theta_1p_1-1)n_1}2^{(\theta_2p_2-1)n_2}2^{(\theta_3\alpha-1)n_3} \quad + \\
&\quad \sum_{n_1,n_2>-3k,\ -N<n_3\le 0} 2^{(\theta_1p_1-1)n_1}2^{(\theta_2p_2-1)n_2}2^{(\theta_3\alpha-1)n_3} \\
&= A + B.
\end{aligned}
$$

For the first term, take $\theta_1 = 1/(2p_1)$, $\theta_2 = 1/(2p_2)$, $\theta_3 = 1-1/(2p)$, and $\alpha = 1$. For the second term, take $\theta_1 = 1/(3p_1)$, $\theta_2 = 1/(3p_2)$, $\theta_3 = 1-1/(3p) > 0$, and $\alpha = 2/\theta_3$ to see

$$A = \sum_{n_1,n_2>-3k,\ n_3>0} 2^{-n_1/2}2^{-n_2/2}2^{-n_3/2p} \lesssim 2^{3k},$$

$$B = \sum_{n_1,n_2>-3k,\ -N<n_3\leq 0} 2^{-2n_1/3}2^{-2n_2/3}2^{n_3} \leq \sum_{n_1,n_2>-3k, n_3\leq 0} 2^{-2n_1/3}2^{-2n_2/3}2^{n_3} \lesssim 2^{4k}.$$

The estimate for A is made in part because p is bounded away from 0 ($p \geq 1/2$). Also, there is no dependence on the number N, which depends on $\mathcal{D}$, or C, which depends on E.

Combining the estimates for $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, we see

$$\sum_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}} |\langle \phi_I^1, f\rangle||\langle \phi_I^2, g\rangle||\langle \phi_I^{3,k}, \chi_{E'}\rangle| \lesssim 2^{4k},$$

where the constant has no dependence on the collection $\mathcal{D}$. Hence, as $\mathcal{D}$ is arbitrary, we have

$$\Big| \sum_I \epsilon_I \frac{1}{|I|^{1/2}} \langle \phi_I^1, f\rangle\langle \phi_I^2, g\rangle\langle \phi_I^{3,k}, \chi_{E'}\rangle \Big| \leq \sum_I \frac{1}{|I|^{1/2}} |\langle \phi_I^1, f\rangle||\langle \phi_I^2, g\rangle||\langle \phi_I^{3,k}, \chi_{E'}\rangle| \lesssim 2^{4k},$$

which completes the proof. □

It should now be clear that proving the above for $a \neq 1$ follows by permuting the roles of M and S. In particular, M will always be applied to the function in the a^{th} slot and S to the others.

For any $\vec{n} \in \mathbb{Z}^2$, we can define the shifted paraproducts by

$$T_\epsilon^{a,[\vec{n}]}(f,g)(x) = \int_0^1 \sum_I \epsilon_I \frac{1}{|I|^{1/2}} \langle \phi^1_{I_\alpha^{n_1}}, f\rangle \langle \phi^2_{I_\alpha^{n_2}}, g\rangle \phi^3_{I_\alpha}(x)\, d\alpha,$$

where, as before, $\int_{\mathbb{T}} \varphi_I^i\, dm = 0$ for $i \neq a$. Much like in Section 5.1, understanding these operators is just a matter of reworking the proof. Simply replace M by $M^{[n_j]}$ and S by $S^{[n_j]}$ where appropriate. This leads to the previous estimates with an additional factor of $(|n_1|+1)(|n_2|+1)$.

5.4 Coifmann-Meyer Operators

We will employ the standard "∂" notation of partial derivatives. That is, $\partial_j^k f$ is the k^{th} partial derivative of f in the j^{th} variable. Further, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a vector of nonnegative integers and $f : \mathbb{R}^d \to \mathbb{C}$, then

$$\partial^\alpha f(\vec{x}) = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f(x_1, \ldots, x_n)$$

For such a vector α, we write $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

Definition. Let $m : \mathbb{R}^d \to \mathbb{C}$ be smooth away from 0 and uniformly bounded. We say m is a Coifman-Meyer multiplier if $|\partial^\alpha m(\vec{t})| \lesssim \|\vec{t}\|^{-|\alpha|}$ for all vectors α with $|\alpha| \le d(d+3)$, where $\|\vec{t}\|$ is the standard Euclidean norm on $\mathbb{R}^d$.

For a Coifman-Meyer multiplier m on $\mathbb{R}^d$ and L^1 functions $f_1, \ldots, f_d : \mathbb{T} \to \mathbb{C}$, we define the multilinear multiplier operator $\Lambda_m(f_1, \ldots, f_d) : \mathbb{T} \to \mathbb{C}$ as

$$\Lambda_m(f_1, \ldots, f_d)(x) = \sum_{\vec{t} \in \mathbb{Z}^d} m(\vec{t}) \widehat{f}_1(t_1) \cdots \widehat{f}_d(t_d) e^{2\pi i x(t_1 + \ldots + t_d)}.$$

The principal goal we have for these operators is the following L^p result.

Theorem. *For any Coifman-Meyer multiplier m on $\mathbb{R}^d$, $\Lambda_m : L^{p_1} \times \ldots \times L^{p_d} \to L^p$ for $1 < p_j < \infty$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_d} = \frac{1}{p}$. If any or all of the p_j are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$.*

For simplicity, we will focus on the $d = 2$ case, but there is no difference in the proof. We start with the following.

Claim 5.6. *Let $f, g, h : \mathbb{T} \to \mathbb{C}$ be smooth. Then,*

$$\sum_{s,t \in \mathbb{Z}} \widehat{f}(s) \widehat{g}(t) \widehat{h}(-s-t) = \int_{\mathbb{T}} f(x) g(x) h(x) \, dx.$$

Proof. As f is smooth, we have the inversion formula $f(x) = \sum_s \widehat{f}(s)e^{2\pi i x s}$. Similarly for g. So,

$$\begin{aligned}\sum_{s,t\in\mathbb{Z}} \widehat{f}(s)\widehat{g}(t)\widehat{h}(-s-t) &= \sum_{s,t\in\mathbb{Z}} \widehat{f}(s)\widehat{g}(t)\Big(\int_{\mathbb{T}} h(x)e^{-2\pi i x(-s-t)}\,dx\Big)\\ &= \int_{\mathbb{T}} h(x)\Big(\sum_{s\in\mathbb{Z}}\widehat{f}(s)e^{2\pi i x s}\Big)\Big(\sum_{t\in\mathbb{Z}}\widehat{g}(t)e^{2\pi i x t}\Big)\,dx\\ &= \int_{\mathbb{T}} f(x)g(x)h(x)\,dx.\end{aligned}$$

□

Lemma 5.7. *Let $m : \mathbb{R}^2 \to \mathbb{C}$ be any Coifmann-Meyer multiplier and $\psi_k^{a,1}, \psi_k^{a,2}$, $a = 1,2,3$, the functions guaranteed by Theorem 1.6. For each $k \in \mathbb{N}$ and $1 \le a \le 3$, there is a smooth function $m_{a,k}$ so that $m_{a,k}(s,t)\widehat{\psi_k^{a,1}}(s)\widehat{\psi_k^{a,2}}(t) = m(s,t)\widehat{\psi_k^{a,1}}(s)\widehat{\psi_k^{a,2}}(t)$ and*

$$m_{a,k}(s,t) = \sum_{\vec{n}\in\mathbb{Z}^2} c_{a,k,\vec{n}}e^{-2\pi i n_1 2^{-k}s}e^{-2\pi i n_2 2^{-k}t},$$

where $|c_{a,k,\vec{n}}| \lesssim (|n_1|+1)^{-5}(|n_2|+1)^{-5}$ uniformly in a and k.

Proof. Let $\varphi_1 : \mathbb{R}^2 \to \mathbb{C}$ be a smooth function with

$$\begin{aligned}\operatorname{supp}(\varphi_1) &\subseteq \Big([-2^{-1},-2^{-11}]\cup[2^{-11},2^{-1}]\Big)\times[2^{-1},2^{-1}] \quad \text{and}\\ \varphi_1 = 1 &\ \text{ on } \Big([-2^{-2},-2^{-10}]\cup[2^{-10},2^{-2}]\Big)\times[-2^{-2},2^{-2}].\end{aligned}$$

Let $\varphi_3 = \varphi_1$ and $\varphi_2(x,y) = \varphi_1(y,x)$. Define $m_{a,k}(s,t) = m(s,t)\varphi_a(2^{-k}s, 2^{-k}t)$. Then, $m_{a,k}(s,t)\widehat{\psi_k^{a,1}}(s)\widehat{\psi_k^{a,2}}(t) = m(s,t)\widehat{\psi_k^{a,1}}(s)\widehat{\psi_k^{a,2}}(t)$ by construction. Further, if $E_{a,k}$ is the support of $m_{a,k}$, then $E_{a,k} \subset [-2^{k-1}, 2^{k-1}]^2$.

Recall that $\{2^{-k/2}e^{-2\pi i n 2^{-k}x}\}$ is an orthonormal basis on any interval of length 2^k, so

$$m_{a,k}(s,t) = \sum_{\vec{n}\in\mathbb{Z}^2} \left(\int_{\mathbb{R}^2} m_{a,k}(x,y) \frac{e^{2\pi i n_1 2^{-k} x}}{2^{k/2}} \frac{e^{2\pi i n_2 2^{-k} y}}{2^{k/2}} \, dx\, dy \right) \frac{e^{-2\pi i n_1 2^{-k} s}}{2^{k/2}} \frac{e^{-2\pi i n_2 2^{-k} t}}{2^{k/2}}$$
$$= \sum_{\vec{n}\in\mathbb{Z}^2} c_{a,k,\vec{n}} e^{-2\pi i n_1 2^{-k} s} e^{-2\pi i n_2 2^{-k} t},$$

where $c_{a,k,\vec{n}} = 2^{-2k} \int_{\mathbb{R}^2} m_{a,k}(x,y) e^{2\pi i n_1 2^{-k} x} e^{2\pi i n_2 2^{-k} y} \, dx\, dy$.

First, if $\vec{n} = (0,0)$, then $c_{a,k,\vec{n}} = 2^{-2k} \int_{\mathbb{R}^2} m_{a,k}\, dm = 2^{-2k} \int_{E_k} m_{a,k}\, dm$. So, $|c_{a,k,\vec{n}}| \le 2^{-2k} |E_k| \|m\|_\infty \|\varphi\|_\infty \le \|m\|_\infty \|\varphi\|_\infty \lesssim 1$.

Assume $n_1 \ne 0$, $n_2 \ne 0$. Let $C = \max\{\|\partial^\alpha \varphi_a\|_\infty : 0 \le |\alpha| \le 10, a = 1,2,3\}$. Note, for $(x,y) \in E_{a,k}$, $|x| \ge 2^{k-11}$ if $a = 1,3$ and $|y| \ge 2^{k-11}$ if $a = 2$. So, $\|(x,y)\| \ge 2^{k-11}$ on $E_{a,k}$ and $|\partial^\alpha m(x,y)| \lesssim \|(x,y)\|^{-|\alpha|} \le |2^{k-11}|^{-|\alpha|} = 2^{-k|\alpha|} 2^{11|\alpha|}$ for all $|\alpha| \le 10$. Set $\beta = (5,5)$. Write $\alpha \le \beta$ if $\alpha_1 \le \beta_1$ and $\alpha_2 \le \beta_2$. Then,

$$|\partial^\beta m_{a,k}(x,y)| \lesssim \sum_{\alpha\le\beta} |\partial^\alpha m(x,y)| |2^{-k(|\beta|-|\alpha|)} \partial^{\beta-\alpha} \varphi(2^{-k}x, 2^{-k}y)|$$
$$\le \sum_{\alpha\le\beta} 2^{-k|\alpha|} 2^{11|\alpha|} 2^{-10k} 2^{k|\alpha|} C \lesssim 2^{-10k}.$$

By several iterations of integration by parts,

$$\left| \int_{\mathbb{R}^2} m_{a,k}(x) e^{2\pi i n_1 2^{-k} x} e^{2\pi i n_2 2^{-k} y} \, dx\, dy \right|$$
$$= \left| \int_{E_{a,k}} m_{a,k}(x) e^{2\pi i n_1 2^{-k} x} e^{2\pi i n_2 2^{-k} y} \, dx\, dy \right|$$
$$= \left| \int_{E_{a,k}} \partial^\beta m_{a,k}(x) \frac{e^{2\pi i n_1 2^{-k} x} e^{2\pi i n_2 2^{-k} y}}{(2\pi i n_1 2^{-k})^5 (2\pi i n_2 2^{-k})^5} \, dx\, dy \right|$$
$$\lesssim \frac{2^{10k}}{|n_1|^5 |n_2|^5} |E_{a,k}| \|\partial^\beta m_{a,k}\|_\infty \lesssim \frac{2^{2k}}{|n_1|^5 |n_2|^5} \lesssim \frac{2^{2k}}{(|n_1|+1)^5 (|n_2|+1)^5}.$$

Namely, $|c_{a,k,\vec{n}}| \lesssim (|n_1|+1)^{-5} (|n_2|+1)^{-5}$. If $n_1 = 0$, repeat the above argument with $\beta = (0,5)$. If $n_2 = 0$, use $\beta = (5,0)$. $\square$

Theorem 5.8. *For any Coifman-Meyer multiplier m on $\mathbb{R}^2$, $\Lambda_m : L^{p_1} \times L^{p_2} \to L^p$ for $1 < p_1, p_2 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. If p_1 or p_2 or both are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$.*

Proof. Fix m and let $f, g : \mathbb{T} \to \mathbb{C}$. Then,

$$\Lambda_m(f,g)(x) = \sum_{s,t \in \mathbb{Z}} m(s,t)\widehat{f}(s)\widehat{g}(t)e^{2\pi i x(s+t)}.$$

As in the proof of Theorem 5.3, we can assume $m(0,0) = 0$, as $|m(0,0)\widehat{f}(0)\widehat{g}(0)| \lesssim \|f\|_1\|g\|_1 \leq \|f\|_{p_1}\|g\|_{p_2}$.

Let $h \in L^1(\mathbb{T})$. Write $f_0 = \widetilde{\widetilde{f}}$ and similarly for g_0, h_0. Then,

$$\begin{aligned}
\langle \Lambda_m(f,g), \widetilde{h} \rangle &= \int_{\mathbb{T}} \Lambda_m(f,g)(x) h_0(x)\, dx \\
&= \int_{\mathbb{T}} \Bigg(\sum_{s,t \in \mathbb{Z}} m(s,t)\widehat{f}(s)\widehat{g}(t)e^{2\pi i x(s+t)} \Bigg) h_0(x)\, dx \\
&= \sum_{s,t \in \mathbb{Z}} m(s,t)\widehat{f}(s)\widehat{g}(t) \int_{\mathbb{T}} h_0(x) e^{2\pi i x(s+t)}\, dx \\
&= \sum_{s,t \in \mathbb{Z}} m(s,t)\widehat{f}(s)\widehat{g}(t)\widehat{h_0}(-s-t).
\end{aligned}$$

Now apply Theorem 1.6 to write

$$\begin{aligned}
\langle \Lambda_m(f,g), \widetilde{h} \rangle &= \sum_{a=1}^{3}\sum_{k=1}^{\infty}\sum_{s,t \in \mathbb{Z}} m(s,t)\widehat{f}(s)\widehat{\psi_k^{a,1}}(s)\widehat{g}(t)\widehat{\psi_k^{a,2}}(t)\widehat{h_0}(-s-t)\widehat{\psi_k^{a,3}}(-s-t) \\
&= \sum_{a=1}^{3}\sum_{k=1}^{\infty}\sum_{s,t \in \mathbb{Z}} m_{a,k}(s,t)\widehat{f}(s)\widehat{\psi_k^{a,1}}(s)\widehat{g}(t)\widehat{\psi_k^{a,2}}(t)\widehat{h_0}(-s-t)\widehat{\psi_k^{a,3}}(-s-t) \\
&=: S_1 + S_2 + S_3,
\end{aligned}$$

where $m_{a,k}$ is as given in Lemma 5.7. Let $\psi_{k,n_1}^{a,1}(x) = \psi_k^{a,1}(x - n_1 2^{-k})$ and $\psi_{k,n_2}^{a,2}(x) = \psi_k^{a,2}(x - n_2 2^{-k})$. Then,

$$
\begin{aligned}
S_a &= \sum_{k=1}^{\infty} \sum_{s,t\in\mathbb{Z}} m_{a,k}(s,t)\widehat{f}(s)\widehat{\psi_k^{a,1}}(s)\widehat{g}(t)\widehat{\psi_k^{a,2}}(t)\widehat{h}_0(-s-t)\widehat{\psi_k^{a,3}}(-s-t) \\
&= \sum_{\vec{n}\in\mathbb{Z}^2} \sum_{k=1}^{\infty} \sum_{s,t\in\mathbb{Z}} c_{a,k,\vec{n}}\widehat{f}(s)\widehat{\psi_{k,n_1}^{a,1}}(s)\widehat{g}(t)\widehat{\psi_{k,n_2}^{a,2}}(t)\widehat{h}_0(-s-t)\widehat{\psi_k^{a,3}}(-s-t) \\
&= \sum_{\vec{n}\in\mathbb{Z}^2} \sum_{k=1}^{\infty} \sum_{s,t\in\mathbb{Z}} c_{a,k,\vec{n}}(f*\psi_{k,n_1}^{a,1})\widehat{\ }(s)(g*\psi_{k,n_2}^{a,2})\widehat{\ }(t)(h_0*\psi_k^{a,3})\widehat{\ }(-s-t) \\
&= \sum_{\vec{n}\in\mathbb{Z}^2} \sum_{k=1}^{\infty} c_{a,k,\vec{n}} \int_{\mathbb{T}} (f*\psi_{k,n_1}^{a,1})(x)(g*\psi_{k,n_2}^{a,2})(x)(h_0*\psi_k^{a,3})(x)\,dx,
\end{aligned}
$$

where the last line is the application of Claim 5.6. Even though f, g, h_0 are not necessarily smooth, their convolutions with smooth functions will be. Just as in the proof of Theorem 5.3, we can dilate and translate to write

$$
\begin{aligned}
&\int_{\mathbb{T}} (f*\psi_{k,n_1}^{a,1})(x)(g*\psi_{k,n_2}^{a,2})(x)(h_0*\psi_k^{a,3})(x)\,dx \\
&\quad = 2^{-k}\int_0^{2^k} (f*\psi_{k,n_1}^{a,1})(2^{-k}x)(g*\psi_{k,n_2}^{a,2})(2^{-k}x)(h_0*\psi_k^{a,3})(2^{-k}x)\,dx \\
&\quad = 2^{-k}\sum_{j=0}^{2^k-1}\int_0^1 \langle \psi_{k,j,n_1,\alpha}^{a,1}, \overline{f}\rangle\langle \psi_{k,j,n_2,\alpha}^{a,2}, \overline{g}\rangle\langle \psi_{k,j,\alpha}^{a,3}, \overline{h_0}\rangle\,d\alpha,
\end{aligned}
$$

where $\psi_{k,j,n_1,\alpha}^{a,1}(x) = \psi_{k,n_1}^{a,1}(2^{-k}(\alpha+j)-x) = \psi_k^{a,1}(2^{-k}(\alpha+j+n_1)-x)$, and similarly for the other two functions.

For a dyadic interval $I = [2^{-k}j, 2^{-k}(j+1)]$, let $\varphi_{I_\alpha^{n_1}}^{a,1} = 2^{-k}\widetilde{\psi_{k,j,n_1,\alpha}^{a,1}}$, $\varphi_{I_\alpha^{n_2}}^{a,2} = 2^{-k}\widetilde{\psi_{k,j,n_2,\alpha}^{a,2}}$, and $\varphi_{I_\alpha}^{a,3} = 2^{-k}\widetilde{\psi_{k,j,\alpha}^{a,3}}$. It is easily checked that the original conditions on $\psi^{a,i}$ guarantee that $\varphi_I^{a,i}$ are adapted families with mean 0 when $a \neq i$. Let $\phi_I^{a,i} = |I|^{-1/2}\varphi_I^{a,i}$, so that

$$
\begin{aligned}
S_a &= \sum_{\vec{n}\in\mathbb{Z}^2}\sum_{k=1}^{\infty} c_{a,k,\vec{n}} 2^{-k}\sum_{j=0}^{2^k-1}\int_0^1 \langle \psi_{k,j,n_1,\alpha}^{a,1}, \overline{f}\rangle\langle \psi_{k,j,n_2,\alpha}^{a,2}, \overline{g}\rangle\langle \psi_{k,j,\alpha}^{a,3}, \overline{h_0}\rangle\,d\alpha \\
&= \sum_{\vec{n}\in\mathbb{Z}^2}\int_0^1 \sum_I c_{a,I,\vec{n}}\frac{1}{|I|^{1/2}}\langle \phi_{I_\alpha^{n_1}}^{a,1}, f_0\rangle\langle \phi_{I_\alpha^{n_2}}^{a,2}, g_0\rangle\langle \phi_{I_\alpha}^{a,3}, h\rangle\,d\alpha,
\end{aligned}
$$

where the inner sum is over all dyadic intervals and $c_{a,I,\vec{n}} = c_{a,k,\vec{n}}$ when $|I| = 2^{-k}$. Write $c'_{a,I,\vec{n}} = (|n_1|+1)^5(|n_2|+1)^5 c_{a,I,\vec{n}}$, which are uniformly bounded in I and $\vec{n}$ by Lemma 5.7. Hence,

$$\begin{aligned} S_a &= \sum_{\vec{n}\in\mathbb{Z}^2} \frac{1}{(|n_1|+1)^5(|n_2|+1)^5} \int_0^1 \sum_I c'_{a,I,\vec{n}} \frac{1}{|I|^{1/2}} \langle \phi^{a,1}_{I^{n_1}_\alpha} f_0\rangle \langle \phi^{a,2}_{I^{n_2}_\alpha}, g_0\rangle \langle \phi^{a,3}_{I_\alpha}, h\rangle \, d\alpha \\ &= \sum_{\vec{n}\in\mathbb{Z}^2} \frac{1}{(|n_1|+1)^5(|n_2|+1)^5} \langle T^{a,[\vec{n}]}_{c'}(f_0, g_0), h\rangle \\ &= \Big\langle \sum_{\vec{n}\in\mathbb{Z}^2} \frac{1}{(|n_1|+1)^5(|n_2|+1)^5} T^{a,[\vec{n}]}_{c'}(f_0, g_0), h\Big\rangle \end{aligned}$$

As $h \in L^1$ is arbitrary, it follows that

$$\widetilde{\Lambda_m(f,g)} = \sum_{\vec{n}\in\mathbb{Z}^2} \frac{1}{(|n_1|+1)^5(|n_2|+1)^5} \sum_{a=1}^{3} T^{a,[\vec{n}]}_{c'}(f_0, g_0)$$

almost everywhere. We know $\|T^{a,[\vec{n}]}_{c'}(f_0,g_0)\|_p \lesssim (|n_1|+1)(|n_2|+1)\|f\|_{p_1}\|g\|_{p_2}$ when $p_1, p_2 > 1$, and $\|T^{a,[\vec{n}]}_{c'}(f_0,g_0)\|_{p,\infty} \lesssim (|n_1|+1)(|n_2|+1)\|f\|_{p_1}\|g\|_{p_2}$ when p_1 or p_2 or both are equal to 1. So, $\|\Lambda_m(f,g)\|_p \lesssim \|f\|_{p_1}\|g\|_{p_2}$ whenever $p \geq 1$, $p_1, p_2 > 1$ follows immediately. By Lemma 5.1 (with $k = 2$), $\|\Lambda_m(f,g)\|_{p,\infty} \lesssim \|f\|_{p_1}\|g\|_{p_2}$ for all $p_1, p_2 \geq 1$; the sum over a does not cause any problems. By interpolation of these cases, $\|\Lambda_m(f,g)\|_p \lesssim \|f\|_{p_1}\|g\|_{p_2}$ whenever $p_1, p_2 > 1$ and $p < 1$. $\square$

Chapter 6
Bi-parameter Multipliers

6.1 Hybrid Max-Square Functions

When considering bi-parameter multipliers, the max and square functions of previous chapters can no longer be applied. However, they can be properly extended to this setting [26, 27].

We say a set $R \subset \mathbb{T}^2$ is a dyadic rectangle if there exist dyadic intervals I and J so that $R = I \times J$. Given two adapted families φ_I^1 and φ_I^2, we will write $\varphi_R(x,y) = \varphi_I^1(x)\varphi_J^2(y)$ for $R = I \times J$. We will informally write $\{\varphi_R\}$ to mean the collection over all dyadic rectangles R. For $\varphi_R = \varphi_I^1 \oplus \varphi_J^2$, set $\phi_R = |R|^{-1/2}\varphi_R = \phi_I^1 \oplus \phi_J^2$.

For functions $f : \mathbb{T}^2 \to \mathbb{C}$, define

$$MMf(x,y) = \sup_R \frac{1}{|R|^{1/2}} |\langle \phi_R, f\rangle| \chi_R(x,y).$$

If $\{\varphi_R\}$ is a family such that $\int_{\mathbb{T}} \varphi_J^2 \, dm = 0$ for all J, then define

$$MSf(x,y) = \sup_I \frac{1}{|I|^{1/2}} \left(\sum_J \frac{|\langle \phi_R, f\rangle|^2}{|J|} \chi_J(y) \right)^{1/2} \chi_I(x),$$

where of course $R = I \times J$. This MS operator is similar to taking a square function S of f in the its second variable, then a maximal function M' in its first variable. Analogously, if $\int_{\mathbb{T}} \varphi_I^1 \, dm = 0$ for all I, define

$$SMf(x,y) = \left(\sum_I \frac{\left(\sup_J \frac{1}{|J|^{1/2}} |\langle \phi_R, f\rangle| \chi_J(y) \right)^2}{|I|} \chi_I(x) \right)^{1/2}.$$

Finally, if $\int_{\mathbb{T}} \varphi_I^1 \, dm = \int_{\mathbb{T}} \varphi_J^2 \, dm = 0$, set

$$SSf(x,y) = \left(\sum_R \frac{|\langle \phi_R, f\rangle|^2}{|R|} \chi_R(x,y) \right)^{1/2}.$$

We note that the "M" in MS, SM, and MM really corresponds to an M'. However, this should not cause any confusion.

From now on, we will be less rigid about the notation. If we write ϕ_R, it will be understood to be a collection over all dyadic rectangles, where each $\phi_R = \phi_I^1 \oplus \phi_J^2$. Further, whenever we employ MM, SM, MS, or SS, it will be understood that there are underlying adapted families and they have integral 0 in the appropriate variable.

Theorem 6.1. *Each of MM, MS, SM, and SS maps $L^p(\mathbb{T}^2) \to L^p(\mathbb{T}^2)$ for $1 < p < \infty$ and $L\log L(\mathbb{T}^2) \to L^{1,\infty}(\mathbb{T}^2)$.*

Proof. Throughout this proof, we will write $\phi_R = \phi_I \oplus \phi_J$, instead of ϕ_I^1 and ϕ_J^2. This is simply for neatness. The underlying adapted families can still be distinct. Recall the notation L_j from Section 2.3. We apply this to M, M', and S. In particular, M_1, M_2, M_1', M_2', S_1, and S_2 each map $L^p(\mathbb{T}^2) \to L^p(\mathbb{T}^2)$ for $1 < p < \infty$, $L^1(\mathbb{T}^2) \to L^{1,\infty}(\mathbb{T}^2)$, and $L\log L(\mathbb{T}^2) \to L^1(\mathbb{T}^2)$ by interpolation. Further, each satisfies Fefferman-Stein inequalities for $r = 2$.

Use Theorem 1.9 to write

$$\varphi_R = \varphi_I \oplus \varphi_J = \left(\sum_{k_1=1}^{\infty} 2^{-10k_1}\varphi_I^{k_1}\right) \oplus \left(\sum_{k_2=1}^{\infty} 2^{-10k_2}\varphi_J^{k_2}\right) =: \sum_{\vec{k}\in\mathbb{N}^2} 2^{-10|\vec{k}|}\varphi_R^{\vec{k}}$$

where each $\varphi_R^{\vec{k}}$ is the tensor product of functions uniformally adapted to I, J respectively. We write $|\vec{k}| = k_1 + k_2$. If each k_1, k_2 is small enough, $\operatorname{supp}(\varphi_R^{\vec{k}}) \subseteq 2^{\vec{k}}R := 2^{k_1}I \times 2^{k_2}J$. Otherwise, $\varphi_R^{\vec{k}}$ is identically 0. Let $K(R)$ be the subset of $\mathbb{N}^2$ for which the first case occurs. As they are uniformally adapted, $\|\varphi_R^{\vec{k}}\|_\infty \lesssim 1$ uniformly in $\vec{k}$ and R. Fix R and suppose $(x, y) \in R$. Then,

$$
\begin{aligned}
\frac{1}{|R|}|\langle\varphi_R,f\rangle|\chi_R(x,y) &\le \frac{1}{|R|}\sum_{\vec{k}\in\mathbb{N}^2} 2^{-10|\vec{k}|}\int_{\mathbb{T}^2}|\varphi_R^{\vec{k}}||f|\,dm \\
&= \frac{1}{|R|}\sum_{\vec{k}\in K(R)} 2^{-10|\vec{k}|}\int_{2^{\vec{k}}R}|\varphi_R^{\vec{k}}||f|\,dm \\
&\lesssim \sum_{\vec{k}\in K(R)} 2^{-9|\vec{k}|}\frac{1}{|2^{\vec{k}}R|}\int_{2^{\vec{k}}R}|f|\,dm \\
&\le \sum_{\vec{k}\in\mathbb{N}^2} 2^{-9|\vec{k}|}M_Sf(x,y) \lesssim M_Sf(x,y).
\end{aligned}
$$

If (x,y) is not in R, then this inequality holds trivially. As R is arbitrary, $MMf \lesssim M_Sf \le M_1\circ M_2 f$. Hence,

$$
\begin{aligned}
\|MMf\|_p &\lesssim \|M_1\circ M_2f\|_p \lesssim \|M_2f\|_p \lesssim \|f\|_p, \\
\|MMf\|_{1,\infty} &\lesssim \|M_1\circ M_2f\|_{1,\infty} \lesssim \|M_2f\|_1 \lesssim \|f\|_{L\log L}.
\end{aligned}
$$

We abuse notation slightly and write $\langle f,\phi_I\rangle$ to mean $\int_{\mathbb{T}}\overline{\phi}_I(x)f(x,y)\,dx$, a function of the variable y. Thus, $\langle\phi_R,f\rangle = \langle\phi_J,\langle f,\phi_I\rangle\rangle$ makes sense. Also, we can consider the two variable function $\langle f,\phi_I\rangle\chi_I$. In this manner,

$$
\begin{aligned}
SMf(x,y) &= \Bigg(\sum_I \frac{\Big(\sup_J \frac{1}{|J|^{1/2}}|\langle\phi_R,f\rangle|\chi_J(y)\Big)^2}{|I|}\chi_I(x)\Bigg)^{1/2} \\
&= \Bigg(\sum_I\Big(\sup_J \frac{1}{|J|^{1/2}}\Big|\Big\langle\phi_J,\frac{\langle f,\phi_I\rangle}{|I|^{1/2}}\chi_I(x)\Big\rangle\Big|\chi_J(y)\Big)^2\Bigg)^{1/2} \\
&= \Bigg(\sum_I M_2'\Big(\frac{\langle f,\phi_I\rangle}{|I|^{1/2}}\chi_I\Big)(x,y)^2\Bigg)^{1/2}.
\end{aligned}
$$

By the Fefferman-Stein inequalities on M' (or M_2'),

$$
\begin{aligned}
\|SMf\|_p &= \Bigg\|\Bigg(\sum_I M_2'\Big(\frac{\langle f,\phi_I\rangle}{|I|^{1/2}}\chi_I\Big)^2\Bigg)^{1/2}\Bigg\|_p \\
&\lesssim \Bigg\|\Bigg(\sum_I \frac{|\langle f,\phi_I\rangle|^2}{|I|}\chi_I\Bigg)^{1/2}\Bigg\|_p = \|S_1f\|_p \lesssim \|f\|_p,
\end{aligned}
$$

and

$$\begin{aligned}\|SMf\|_{1,\infty} &= \left\| \left(\sum_I M_2'\Big(\frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I \Big)^2 \right)^{1/2} \right\|_{1,\infty} \\ &\lesssim \left\| \left(\sum_I \frac{|\langle f, \phi_I\rangle|^2}{|I|} \chi_I \right)^{1/2} \right\|_1 = \|S_1 f\|_1 \lesssim \|f\|_{L\log L}.\end{aligned}$$

On the other hand,

$$\begin{aligned}MSf(x,y) &= \sup_I \frac{1}{|I|^{1/2}} \left(\sum_J \frac{|\langle \phi_R, f\rangle|^2}{|J|} \chi_J(y) \right)^{1/2} \chi_I(x) \\ &\leq \left(\sum_J \frac{\Big(\sup_I \frac{1}{|I|^{1/2}} |\langle \phi_R, f\rangle| \chi_I(x) \Big)^2}{|J|} \chi_J(y) \right)^{1/2}.\end{aligned}$$

This is essentially SM with the roles of I and J reversed. The same arguments as above can now be applied.

Finally,

$$\begin{aligned}SSf(x,y) &= \left(\sum_R \frac{|\langle \phi_R, f\rangle|^2}{|R|} \chi_R(x,y) \right)^{1/2} \\ &= \left[\sum_I \sum_J \frac{1}{|J|} \Big| \Big\langle \phi_J, \frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I(x) \Big\rangle \Big|^2 \chi_J(y) \right]^{1/2} \\ &= \left[\sum_I S_2\Big(\frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I \Big)(x,y)^2 \right]^{1/2},\end{aligned}$$

so that by the Fefferman-Stein inequalities on S_2,

$$\begin{aligned}\|SSf\|_p &= \left\| \left(\sum_I S_2\Big(\frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I \Big)^2 \right)^{1/2} \right\|_p \\ &\lesssim \left\| \left(\sum_I \frac{|\langle f, \phi_I\rangle|^2}{|I|} \chi_I \right)^{1/2} \right\|_p = \|S_1 f\|_p \lesssim \|f\|_p,\end{aligned}$$

and

$$\|SSf\|_{1,\infty} = \left\| \left(\sum_I S_2\Big(\frac{\langle f, \phi_I\rangle}{|I|^{1/2}}\chi_I\Big)^2 \right)^{1/2} \right\|_{1,\infty}$$
$$\lesssim \left\| \left(\sum_I \frac{|\langle f, \phi_I\rangle|^2}{|I|}\chi_I \right)^{1/2} \right\|_1 = \|S_1 f\|_1 \lesssim \|f\|_{L\log L}.$$

□

Let $R = I \times J$ be a dyadic rectangle. For $\vec{n} \in \mathbb{Z}^2$ and $\vec{\alpha} \in [0,1]^2$, let $R^{\vec{n}}_{\vec{\alpha}} = I^{n_1}_{\alpha_1} \times J^{n_2}_{\alpha_2}$ and $\varphi_{R^{\vec{n}}_{\vec{\alpha}}} = \varphi_{I^{n_1}_{\alpha_1}} \oplus \varphi_{J^{n_2}_{\alpha_2}}$. In this way, we can define shifted versions of each of MM, SM, MS, and SS. For example,

$$SS^{\vec{n}}_{\vec{\alpha}} f(x,y) = \left(\sum_R \frac{|\langle \phi_{R^{\vec{n}}_{\vec{\alpha}}}, f\rangle|^2}{|R|}\chi_R(x,y) \right)^{1/2},$$

and $SS^{[\vec{n}]}f(x,y) = \sup_{\vec{\alpha}} SS^{\vec{n}}_{\vec{\alpha}} f(x,y)$. We first note that $SS^{\vec{n}}$ satisfies all the above properties with an additional factor of $(|n_1|+1)(|n_2|+1)$. This follows easily by replacing in the previous proof S_1, S_2 by $S_1^{n_1}, S_2^{n_2}$. Then, as before, we observe that $SS^{[\vec{n}]}f$ is bounded by an $SS^{\vec{n}}f$, with a particular adapted tensor product which depends on f. So, $SS^{[\vec{n}]}$ satisfies the above with the additional factor of $(|n_1|+1)(|n_2|+1)$. The same holds for $SM^{[\vec{n}]}$, $MS^{[\vec{n}]}$, and $MM^{[\vec{n}]}$.

Although we will not explicitly need the following result, it is interesting enough to mention here.

Theorem 6.2. *Each of MM, MS, SM, and SS maps $L(\log L)^{n+2} \to L(\log L)^n$.*

Proof. This is simply a matter of repeating the arguments of the previous proof and applying the interpolation results of Corollaries 4.13 and 4.19. We have immediately that $\|MMf\|_{L(\log L)^n} \lesssim \|M_1 \circ M_2 f\|_{L(\log L)^n} \lesssim \|M_2 f\|_{L(\log L)^{n+1}} \lesssim \|f\|_{L(\log L)^{n+2}}$. Further,

$$
\begin{aligned}
\|SMf\|_{L(\log L)^n} &= \left\| \left(\sum_I M_2' \Big(\frac{\langle f, \phi_I \rangle}{|I|^{1/2}} \chi_I \Big)^2 \right)^{1/2} \right\|_{L(\log L)^n} \\
&\lesssim \left\| \left(\sum_I \frac{|\langle f, \phi_I \rangle|^2}{|I|} \chi_I \right)^{1/2} \right\|_{L(\log L)^{n+1}} \\
&= \|S_1 f\|_{L(\log L)^{n+1}} \lesssim \|f\|_{L(\log L)^{n+2}},
\end{aligned}
$$

and

$$
\begin{aligned}
\|SSf\|_{L(\log L)^n} &= \left\| \left(\sum_I S_2 \Big(\frac{\langle f, \phi_I \rangle}{|I|^{1/2}} \chi_I \Big)^2 \right)^{1/2} \right\|_{L(\log L)^n} \\
&\lesssim \left\| \left(\sum_I \frac{|\langle f, \phi_I \rangle|^2}{|I|} \chi_I \right)^{1/2} \right\|_{L(\log L)^{n+1}} \\
&= \|S_1 f\|_{L(\log L)^{n+1}} \lesssim \|f\|_{L(\log L)^{n+2}}.
\end{aligned}
$$

Finally, MS is pointwise smaller than an SM type operator, and therefore satisfies the same bounds. $\square$

6.2 Bi-parameter Paraproducts

In Section 5.3, we defined single-parameter paraproducts. In order to study bi-parameter multiplier operators, we will need to define and investigate the appropriate bi-parameter paraproducts. For simplicity, as before, we will focus only on the bilinear case.

For $f, g : \mathbb{T}^2 \to \mathbb{C}$, the bi-parameter bilinear paraproducts are defined

$$
T_\epsilon^{a,b}(f,g)(x,y) = \sum_R \epsilon_R \frac{1}{|R|^{1/2}} \langle \phi_R^1, f \rangle \langle \phi_R^2, g \rangle \phi_R^3(x,y),
$$

for $a, b = 1, 2, 3$, where φ_R^1, φ_R^2, and φ_R^3 are each the tensor product of two adapted families, as in the previous section. The sum is over all dyadic rectangles R, and

(ϵ_R) is a uniformly bounded sequence. By dividing out a constant, we can assume $|\epsilon_R| \leq 1$. Further, if $\phi_R^i = \phi_I^i \oplus \phi_J^i$, then $\int_{\mathbb{T}} \phi_I^i \, dx = 0$ for $i \neq a$ and $\int_{\mathbb{T}} \phi_J^i \, dx = 0$ for $i \neq b$.

Theorem 6.3. *$T_\epsilon^{a,b} : L^{p_1} \times L^{p_2} \to L^p$ for $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. If p_1 or p_2 or both are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$ and L^{p_j} replaced by $L \log L$. The underlying constants do not depend on a, b, or the sequence ϵ_R.*

Proof. We will assume $a = 1$ and $b = 2$, as the other cases will follow similarly.

First, suppose $p > 1$. Then, necessarily $p_1, p_2 > 1$ and $1 < p' < \infty$. Note, $1/p_1 + 1/p_2 + 1/p' = 1$. Fix $h \in L^{p'}(\mathbb{T})$ with $\|h\|_{p'} \leq 1$. Then,

$$\begin{aligned}
|\langle T_\epsilon^{1,2}(f,g), h\rangle| &= \Big| \sum_R \epsilon_R \frac{1}{|R|^{1/2}} \langle \phi_R^1, f\rangle \langle \phi_R^2, g\rangle \langle \phi_R^3, h\rangle \Big| \\
&\leq \sum_R \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f\rangle| |\langle \phi_R^2, g\rangle| |\langle \phi_R^3, h\rangle| \\
&= \int_{\mathbb{T}^2} \sum_R \frac{|\langle \phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^2, g\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^3, h\rangle|}{|R|^{1/2}} \chi_R(x,y) \, dx \, dy.
\end{aligned}$$

Concentrating on the integrand,

$$\sum_R \frac{|\langle \phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^2, g\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^3, h\rangle|}{|R|^{1/2}} \chi_R(x,y) =$$
$$\sum_I \sum_J \frac{|\langle \phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^2, g\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^3, h\rangle|}{|R|^{1/2}} \chi_R(x,y) \leq$$
$$\sum_I \left[\left(\frac{1}{|I|^{1/2}} \chi_I(x) \sup_J \frac{|\langle \phi_R^2, g\rangle|}{|J|^{1/2}} \chi_J(y) \right) \times \left(\sum_J \frac{|\langle \phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^3, h\rangle|}{|R|^{1/2}} \chi_R(x,y) \right) \right].$$

Applying Hölder's inequality, the last term is bounded by

$$SM(g)(x,y) \left(\sum_I \left(\sum_J \frac{|\langle \phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^3, h\rangle|}{|R|^{1/2}} \chi_R(x,y) \right)^2 \right)^{1/2}.$$

Applying Hölder to the inner sum,

$$\Big(\sum_I \Big(\sum_J \frac{|\langle\phi_R^1, f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^3, h\rangle|}{|R|^{1/2}}\chi_R(x,y)\Big)^2\Big)^{1/2} \leq$$

$$\Big(\sum_I \Big(\sum_J \frac{|\langle\phi_R^1, f\rangle|^2}{|R|}\chi_R(x,y)\Big)\Big(\sum_J \frac{|\langle\phi_R^3, h\rangle|^2}{|R|}\chi_R(x,y)\Big)\Big)^{1/2} \leq$$

$$\Big(\sup_I \frac{1}{|I|}\chi_I(x)\sum_J \frac{|\langle\phi_R^1, f\rangle|^2}{|J|}\chi_J(y)\Big)^{1/2}\Big(\sum_I\sum_J \frac{|\langle\phi_R^3, h\rangle|^2}{|R|}\chi_R(x,y)\Big)^{1/2} =$$

$$MS(f)(x,y)SS(h)(x,y).$$

Hence,

$$\begin{aligned}|\langle T_\epsilon^{1,2}(f,g), h\rangle| &\leq \int_{\mathbb{T}^2} MSf(x,y)SMg(x,y)SSh(x,y)\,dx\,dy \\ &\leq \|MSf\|_{p_1}\|SMg\|_{p_2}\|SSh\|_{p'} \lesssim \|f\|_{p_1}\|g\|_{p_2}.\end{aligned}$$

As h in the unit ball of $L^{p'}$ is arbitrary, we have $\|T_\epsilon^{1,2}(f,g)\|_p \lesssim \|f\|_{p_1}\|g\|_{p_2}$.

Now suppose $1/2 \leq p \leq 1$. As in the proof of Theorem 5.5, by interpolation it suffices to show $T_\epsilon^{1,2} : L^{p_1} \times L^{p_2} \to L^{p,\infty}$ for all $1 < p_1, p_2 < \infty$. We concentrate on the special case $T_\epsilon^{1,2} : L\log L \times L\log L \to L^{1/2,\infty}$, but all others follow in the same way.

Let $\|f\|_{L\log L} = \|g\|_{L\log L} = 1$ and $E \subseteq \mathbb{T}^2$ with $|E| > 0$. Lemma 3.9 is valid on $\mathbb{T}^d$ for any dimension d. So, we will be done if we can find $E' \subseteq E$, $|E'| > |E|/2$ so that $|\langle T_\epsilon^{1,2}(f,g), \chi_{E'}\rangle| \lesssim 1 \leq |E|^{-1}$.

For $\vec{k} \in \mathbb{N}^2$ and $R = I \times J$ a dyadic interval, denote $2^{\vec{k}}R = 2^{k_1}I \times 2^{k_2}J$, and $|\vec{k}| = k_1 + k_2$. Use Theorem 1.10 to write

$$\phi_R^3 = \sum_{\vec{k}\in\mathbb{N}^2} 2^{-10|\vec{k}|}\phi_R^{3,\vec{k}}$$

where each $\phi_R^{3,\vec{k}}$ is the normalization of the tensor product of two 0-mean adapted families which are uniformally adapted to I, J respectively. Further, $\mathrm{supp}(\phi_R^{3,\vec{k}}) \subseteq 2^{\vec{k}}R$ for $\vec{k}$ small enough, while $\phi_I^{3,\vec{k}}$ is identically 0 otherwise. Now

$$\langle T_\epsilon^{1,2}(f,g), \chi_{E'}\rangle = \sum_{\vec{k}\in\mathbb{N}^2} 2^{-10|\vec{k}|} \sum_R \epsilon_R \frac{1}{|R|^{1/2}} \langle \phi_R^1, f\rangle \langle \phi_R^2, g\rangle \langle \phi_R^{3,\vec{k}}, \chi_{E'}\rangle.$$

Hence, it suffices to show $|\sum \epsilon_R |R|^{-1/2} \langle \phi_R^1, f\rangle \langle \phi_R^2, g\rangle \langle \phi_R^{3,\vec{k}}, \chi_{E'}\rangle| \lesssim 2^{4|\vec{k}|}$, so long as the underlying constants are independent of $\vec{k}$.

Let $SS^{\vec{k}}$ be the double square operator with $\phi_R^{\vec{k}}$. For each $\vec{k}\in\mathbb{N}^2$, define

$$\begin{aligned}
\Omega_{-3|\vec{k}|} &= \{MSf > C2^{3|\vec{k}|}\} \cup \{SMg > C2^{3|\vec{k}|}\},\\
\widetilde{\Omega}_{\vec{k}} &= \{M_S(\chi_{\Omega_{-3|\vec{k}|}}) > 1/100\},\\
\widetilde{\widetilde{\Omega}}_{\vec{k}} &= \{M_S(\chi_{\widetilde{\Omega}_{\vec{k}}}) > 2^{-|\vec{k}|-1}\}.
\end{aligned}$$

and

$$\Omega = \bigcup_{\vec{k}\in\mathbb{N}^2} \widetilde{\widetilde{\Omega}}_{\vec{k}}.$$

Observe,

$$|\Omega| \le \sum_{\vec{k}\in\mathbb{N}^2} 2^{-3|\vec{k}|} 2^{2|\vec{k}|+1} \frac{100^2}{C} \|M_S\|_{L^2\to L^2}^4 \Big[\|MS\|_{L\log L\to L^{1,\infty}} + \|SM\|_{L\log L\to L^{1,\infty}}\Big].$$

Therefore, we can choose C independent of f and g so that $|\Omega| < |E|/2$. Set $E' = E - \Omega = E\cap\Omega^c$. Then, $E'\subseteq E$ and $|E'| > |E|/2$.

Fix $\vec{k}\in\mathbb{N}^2$. Set $Z_{\vec{k}} = \{MSf = 0\} \cup \{SMg = 0\} \cup \{SS^{\vec{k}}\chi_{E'} = 0\}$. Let $\mathcal{D}$ be any finite collection of dyadic rectangles. We divide this collection into three subcollections. Set $\mathcal{D}_1 = \{R \in \mathcal{D} : R \cap Z_{\vec{k}} \neq \emptyset\}$. For the remaining rectangles, let $\mathcal{D}_2 = \{R \in \mathcal{D} - \mathcal{D}_1 : R \subseteq \widetilde{\Omega}_{\vec{k}}\}$ and $\mathcal{D}_3 = \{R \in \mathcal{D} - \mathcal{D}_1 : R \cap \widetilde{\Omega}^c_{\vec{k}} \neq \emptyset\}$.

If $R \in \mathcal{D}_1$, then there is some $(x, y) \in R \cap Z_{\vec{k}}$. Namely, $MSf(x, y) = 0$, $SMg(x, y) = 0$, or $SS^{\vec{k}}(\chi_{E'})(x, y) = 0$. If it is the first, $\langle \phi^1_R, f\rangle = 0$. If it is the second, then $\langle \phi^2_R, g\rangle = 0$, and if it is the third, $\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle = 0$. As this holds for all $R \in \mathcal{D}_1$, we have

$$\sum_{R\in\mathcal{D}_1} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f\rangle||\langle \phi^2_R, g\rangle||\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle| = 0.$$

Now suppose $R \in \mathcal{D}_2$, namely $R \subseteq \widetilde{\Omega}_{\vec{k}}$. For some $\vec{k}$, $\phi^{3,\vec{k}}_R$ is identically 0 and $\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle = 0$. For all others, $\phi^{3,\vec{k}}_I$ is supported in $2^{\vec{k}}R$. Let $(x, y) \in 2^{\vec{k}}R$, and observe

$$M_S(\chi_{\widetilde{\Omega}_{\vec{k}}})(x, y) \geq \frac{1}{|2^{\vec{k}}R|} \int_{2^{\vec{k}}R} \chi_{\widetilde{\Omega}_{\vec{k}}}\, dm \geq \frac{1}{2^{|\vec{k}|}} \frac{1}{|R|} \int_R \chi_{\widetilde{\Omega}_{\vec{k}}}\, dm = 2^{-|\vec{k}|} > 2^{-|\vec{k}|-1}.$$

That is, $2^{\vec{k}}R \subseteq \widetilde{\widetilde{\Omega}}_{\vec{k}} \subseteq \Omega$, a set disjoint from E'. Thus, $\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle = 0$. As this holds for all $R \in \mathcal{D}_2$, we have

$$\sum_{R\in\mathcal{D}_2} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f\rangle||\langle \phi^2_R, g\rangle||\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle| = 0.$$

Finally, we concentrate on $\mathcal{D}_3$. Define $\Omega_{-3|\vec{k}|+1}$ and $\Pi_{-3|\vec{k}|+1}$ by

$$\Omega_{-3|\vec{k}|+1} = \{MSf > C2^{3|\vec{k}|-1}\},$$

$$\Pi_{-3|\vec{k}|+1} = \{I \in \mathcal{D}_3 : |I \cap \Omega_{-3|\vec{k}|+1}| > |R|/100\}.$$

Inductively, define for all $n > -3|\vec{k}| + 1$,

$$\Omega_n = \{MSf > C2^{-n}\},$$

$$\Pi_n = \{R \in \mathcal{D}_3 - \bigcup_{j=-3|\vec{k}|+1}^{n-1} \Pi_j : |R \cap \Omega_n| > |R|/100\}.$$

As every $R \in \mathcal{D}_3$ is not in $\mathcal{D}_1$, that is $MSf > 0$ on R, it is clear that each $R \in \mathcal{D}_3$ will be in one of these collections.

Set $\Omega'_{-3|\vec{k}|} = \Omega_{-3|\vec{k}|}$ for symmetry. Define $\Omega'_{-3|\vec{k}|+1}$ and $\Pi'_{-3|\vec{k}|+1}$ by

$$\Omega'_{-3|\vec{k}|+1} = \{SMg > C2^{3|\vec{k}|-1}\},$$

$$\Pi'_{-3|\vec{k}|+1} = \{R \in \mathcal{D}_3 : |R \cap \Omega'_{-3|\vec{k}|+1}| > |R|/100\}.$$

Inductively, define for all $n > -3|\vec{k}| + 1$,

$$\Omega'_n = \{SMg > C2^{-n}\},$$

$$\Pi'_n = \{R \in \mathcal{D}_3 - \bigcup_{j=-3|\vec{k}|+1}^{n-1} \Pi'_j : |R \cap \Omega'_n| > |R|/100\}.$$

As every $R \in \mathcal{D}_3$ is not in $\mathcal{D}_1$, that is $SMg > 0$ on R, it is clear that each $R \in \mathcal{D}_3$ will be in one of these collections.

Now, we can choose an integer N big enough so that $\Omega''_{-N} = \{SS^{\vec{k}}(\chi_{E'}) > 2^N\}$ has very small measure. In particular, we take N big enough so that $|R \cap \Omega''_{-N}| < |R|/100$ for all $R \in \mathcal{D}_3$, which is possible since $\mathcal{D}_3$ is a finite collection. Define

$$\Omega''_{-N+1} = \{SS^{\vec{k}}(\chi_{E'}) > 2^{N-1}\},$$

$$\Pi''_{-N+1} = \{R \in \mathcal{D}_3 : |R \cap \Omega''_{-N+1}| > |R|/100\},$$

and

$$\Omega''_n = \{SS^{\vec{k}}(\chi_{E'}) > 2^{-n}\},$$

$$\Pi''_n = \{R \in \mathcal{D}_3 - \bigcup_{j=-N+1}^{n-1} \Pi''_j : |R \cap \Omega''_n| > |R|/100\},$$

Again, all $R \in \mathcal{D}_3$ must be in one of these collections.

Consider $R \in \mathcal{D}_3$, so that $R \cap \widetilde{\Omega}^c_{\vec{k}} \neq \emptyset$. Then, there is some $(x, y) \in R \cap \widetilde{\Omega}^c_{\vec{k}}$ which implies $|R \cap \Omega_{-3|\vec{k}|}|/|R| \leq M_S(\chi_{\Omega_{-3|\vec{k}|}})(x, y) \leq 1/100$. Write $\Pi_{n_1,n_2,n_3} = \Pi_{n_1} \cap \Pi'_{n_2} \cap \Pi''_{n_3}$. So,

$$\begin{aligned}
&\sum_{R\in\mathcal{D}_3} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f\rangle||\langle \phi^2_R, g\rangle||\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle| \\
&\qquad = \sum_{n_1,n_2>-3|\vec{k}|,\, n_3>-N} \left[\sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f\rangle||\langle \phi^2_R, g\rangle||\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle| \right] \\
&\qquad = \sum_{n_1,n_2>-3|\vec{k}|,\, n_3>-N} \left[\sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{|\langle \phi^1_R, f\rangle|}{|R|^{1/2}} \frac{|\langle \phi^2_R, g\rangle|}{|R|^{1/2}} \frac{|\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle|}{|R|^{1/2}} |R| \right].
\end{aligned}$$

Suppose $R \in \Pi_{n_1,n_2,n_3}$. If $n_1 > -3|\vec{k}| + 1$, then $R \in \Pi_{n_1}$, which in particular says $R \notin \Pi_{n_1-1}$. So, $|R \cap \Omega_{n_1-1}| \leq |R|/100$. If $n_1 = -3|\vec{k}| + 1$, then we still have $|R \cap \Omega_{-3|\vec{k}|}| \leq |R|/100$, as $R \in \mathcal{D}_3$. Similarly, If $n_2 > -3k + 1$, then $R \in \Pi'_{n_2}$, which in particular says $R \notin \Pi'_{n_2-1}$. So, $|R \cap \Omega'_{n_2-1}| \leq |R|/100$. If $n_2 = -3|\vec{k}| + 1$, then we still have $|R \cap \Omega'_{-3|\vec{k}|}| = |R \cap \Omega_{-3|\vec{k}|}| \leq |R|/100$, as $R \in \mathcal{D}_3$. Finally, if $n_3 > -N + 1$, then $R \notin \Pi''_{n_3-1}$ and $|R \cap \Omega''_{n_3-1}| \leq |R|/100$. If $n_3 = -N + 1$, then $|R \cap \Omega''_{-N}| \leq |R|/100$ by the choice of N. So, $|R \cap \Omega^c_{n_1-1} \cap \Omega'^c_{n_2-1} \cap \Omega''^c_{n_3-1}| \geq \frac{97}{100}|R|$. Let $\Omega_{n_1,n_2,n_3} = \bigcup\{R : R \in \Pi_{n_1,n_2,n_3}\}$. Then,

$$|R \cap \Omega^c_{n_1-1} \cap \Omega'^c_{n_2-1} \cap \Omega''^c_{n_3-1} \cap \Omega_{n_1,n_2,n_3}| \geq \frac{97}{100}|R|$$

for all $R \in \Pi_{n_1,n_2,n_3}$. Further,

$$\sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_R^1,f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^2,g\rangle}{|R|^{1/2}}\frac{|\langle\phi_R^{3,\vec{k}},\chi_{E'}\rangle|}{|R|^{1/2}}|R|$$

$$\lesssim \sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_R^1,f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^2,g\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^{3,\vec{k}},\chi_{E'}\rangle|}{|R|^{1/2}}|R\cap\Omega_{n_1-1}^c\cap\Omega_{n_2-1}'^c\cap\Omega_{n_3-1}''^c\cap\Omega_{n_1,n_2,n_3}|$$

$$= \int_{\Omega_{n_1-1}^c\cap\Omega_{n_2-1}'^c\cap\Omega_{n_3-1}''^c\cap\Omega_{n_1,n_2,n_3}} \sum_{I\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_R^1,f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^2,g\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^{3,\vec{k}},\chi_{E'}\rangle|}{|R|^{1/2}}\chi_R\,dm$$

$$\leq \int_{\Omega_{n_1-1}^c\cap\Omega_{n_2-1}'^c\cap\Omega_{n_3-1}''^c\cap\Omega_{n_1,n_2,n_3}} MSf(x,y)SMg(x,y)SS^{\vec{k}}(\chi_{E'})(x,y)\,dx\,dy$$

$$\lesssim C^2 2^{-n_1}2^{-n_2}2^{-n_3}|\Omega_{n_1,n_2,n_3}|.$$

Note, $|\Omega_{n_1,n_2,n_3}| \leq |\bigcup\{R : R\in\Pi_{n_1}\}| \leq |\{M_S(\chi_{\Omega_{n_1}}) > 1/100\}| \lesssim |\Omega_{n_1}| = |\{MSf > C2^{-n_1}\}| \lesssim C^{-1}2^{n_1}$. By the same argument, $|\Omega_{n_1,n_2,n_3}| \lesssim |\Omega'_{n_2}| = |\{SMg > C2^{-n_2}\}| \lesssim C^{-1}2^{n_2}$, and $|\Omega_{n_1,n_2,n_3}| \lesssim |\Omega''_{n_3}| = |\{SS^{\vec{k}}(\chi_{E'}) > 2^{-n_3}\}| \lesssim 2^{\alpha n_3}$ for any $\alpha \geq 1$. Thus, $|\Omega_{n_1,n_2,n_3}| \lesssim C^{-2}2^{\theta_1 n_1}2^{\theta_2 n_2}2^{\theta_3\alpha n_3}$ for any $\theta_1+\theta_2+\theta_3 = 1$, $0 \leq \theta_1,\theta_2,\theta_3 \leq 1$. Hence,

$$\begin{aligned}
&\sum_{R\in\mathcal{D}_3} \frac{1}{|R|^{1/2}}|\langle\phi_R^1,f\rangle||\langle\phi_R^2,g\rangle||\langle\phi_R^{3,\vec{k}},\chi_{E'}\rangle| \\
&\qquad\lesssim \sum_{n_1,n_2>-3|\vec{k}|,\ n_3>0} 2^{(\theta_1-1)n_1}2^{(\theta_2-1)n_2}2^{(\theta_3\alpha-1)n_3} \quad + \\
&\qquad\quad \sum_{n_1,n_2>-3|\vec{k}|,\ -N<n_3\leq 0} 2^{(\theta_1-1)n_1}2^{(\theta_2-1)n_2}2^{(\theta_3\alpha-1)n_3} \\
&\qquad= A+B.
\end{aligned}$$

For the first term, take $\theta_1 = 1/2$, $\theta_2 = 1/2$, $\theta_3 = 0$, and $\alpha = 1$. For the second term, take $\theta_1 = 1/3$, $\theta_2 = 1/3$, $\theta_3 = 1/3$, and $\alpha = 6$ to see

$$A = \sum_{n_1, n_2 > -3|\vec{k}|,\ n_3 > 0} 2^{-n_1/2} 2^{-n_2/2} 2^{-n_3} \lesssim 2^{3|\vec{k}|},$$

$$B = \sum_{n_1, n_2 > -3|\vec{k}|,\ -N < n_3 \leq 0} 2^{-2n_1/3} 2^{-2n_2/3} 2^{n_3}$$

$$\leq \sum_{n_1, n_2 > -3|\vec{k}|, n_3 \leq 0} 2^{-2n_1/3} 2^{-2n_2/3} 2^{n_3} \lesssim 2^{4|\vec{k}|}.$$

Note, there is no dependence on the number N, which depends on $\mathcal{D}$, or C, which depends on E.

Combining the estimates for $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, we see

$$\sum_{R \in \mathcal{D}} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f \rangle| |\langle \phi_R^2, g \rangle| |\langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle| \lesssim 2^{4|\vec{k}|},$$

where the constant has no dependence on the collection $\mathcal{D}$. Hence, as $\mathcal{D}$ is arbitrary, we have

$$\Big| \sum_R \epsilon_R \frac{1}{|R|^{1/2}} \langle \phi_R^1, f \rangle \langle \phi_R^2, g \rangle \langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle \Big|$$
$$\leq \sum_R \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f \rangle| |\langle \phi_R^2, g \rangle| |\langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle| \lesssim 2^{4|\vec{k}|},$$

which completes the proof. □

It should now be clear that proving the above for $(a, b) \neq (1, 2)$ follows by permuting the roles of MM, MS, SM, and SS. For instance, if $(a, b) = (1, 1)$, then we consider MMf, SSg, and $SS^{\vec{k}} \chi_{E'}$.

For any $\vec{n} \in \mathbb{Z}^4$, where $\vec{n}_1 = (n_1, n_2)$ and $\vec{n}_2 = (n_3, n_4)$, we can define the shifted paraproducts by

$$T_\epsilon^{a,b,[\vec{n}]}(f, g)(\vec{x}) = \int_{[0,1]^2} \sum_R \epsilon_R \frac{1}{|R|^{1/2}} \langle \phi^1_{R_{\vec{\alpha}}^{\vec{n}_1}}, f \rangle \langle \phi^2_{R_{\vec{\alpha}}^{\vec{n}_2}}, g \rangle \phi^3_{R_{\vec{\alpha}}}(\vec{x})\, d\vec{\alpha},$$

Like the previous cases, simply rework the proof. For instance, if $(a, b) = (1, 2)$, replace MSf by $MS^{[\vec{n}_1]}f$, SMg by $SM^{[\vec{n}_2]}g$, and $SS^{\vec{k}}(\chi_{E'})$ by $SS^{\vec{k},[0]}(\chi_{E'})$. This leads to the previous estimates with an additional factor of $\prod_{j=1}^{4}(|n_j| + 1)$.

6.3 Multiplier Operators

We now wish to extend Coifman-Meyer operators to a broader bi-parameter setting. In particular, we investigate a new, wider class of multipliers m, which act as if they are the product of two Coifman-Meyer multipliers.

Given a vector $\vec{t} = (t_1, \ldots, t_{2d}) \in \mathbb{R}^{2d}$, denote $\rho_1(\vec{t}) = (t_1, t_3, \ldots, t_{2d-1})$ and $\rho_2(\vec{t}) = (t_2, t_4, \ldots, t_{2d})$, which are both vectors in $\mathbb{R}^d$. For multi-indices of nonnegative integers α, we can also employ this notation. In particular, $|\rho_1(\alpha)| = \alpha_1 + \alpha_3 + \ldots + \alpha_{2d-1}$, and similarly for $\rho_2(\alpha)$. Conversely, for $1 \leq j \leq d$, let $\vec{t_j} = (t_{2j-1}, t_{2j}) \in \mathbb{R}^2$, so that $\vec{t} = (\vec{t_1}, \ldots, \vec{t_d})$.

Definition. Let $m : \mathbb{R}^{2d} \to \mathbb{C}$ be smooth away the origin and uniformly bounded. We say m is a bi-parameter multiplier if $|\partial^\alpha m(\vec{t})| \lesssim \|\rho_1(\vec{t})\|^{-|\rho_1(\alpha)|}\|\rho_2(\vec{t})\|^{-|\rho_2(\alpha)|}$ for all vectors α with $|\alpha| \leq 2d(d + 3)$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$.

Given such a multiplier m on $\mathbb{R}^{2d}$ and L^1 functions $f_1, \ldots, f_d : \mathbb{T}^2 \to \mathbb{C}$, we define the associated multiplier operator $\Lambda_m^{(2)}(f_1, \ldots, f_d) : \mathbb{T}^2 \to \mathbb{C}$ as

$$\Lambda_m^{(2)}(f_1, \ldots, f_d)(\vec{x}) = \sum_{\vec{t} \in \mathbb{Z}^{2d}} m(\vec{t})\widehat{f_1}(\vec{t_1}) \cdots \widehat{f_d}(\vec{t_d})e^{2\pi i\vec{x}\cdot(\vec{t_1}+\ldots+\vec{t_d})}.$$

Consider the following theorem.

Theorem. *For any bi-parameter multiplier m on $\mathbb{R}^{2d}$, $\Lambda_m^{(2)} : L^{p_1} \times \ldots \times L^{p_d} \to L^p$ for $1 < p_j < \infty$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_d} = \frac{1}{p}$. If any or all of the p_j are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$ and L^{p_j} replaced by $L \log L$. In particular, $\Lambda_m^{(2)} : L \log L \times \ldots \times L \log L \to L^{1/d,\infty}$.*

As before, we will focus on the $d = 2$ case for simplicity, but this makes no substantiative difference in the proof. We note that in this case, the bi-parameter multiplier condition can be stated

$$|\partial^{(\alpha,\beta)} m(\vec{s},\vec{t})| \lesssim \|(s_1,t_1)\|^{-\alpha_1-\beta_1}\|(s_2,t_2)\|^{-\alpha_2-\beta_2}$$

for all two-dimensional indices α, β with $|\alpha|, |\beta| \leq 10$.

Remark 6.4. Let $\psi_k^{a,i}$ be the functions in Theorem 1.6. For $1 \leq a, b \leq 3$ and $k, k' \in \mathbb{N}$, define $\psi_{k,k'}^{a,b,i}(\vec{s}) = \psi_k^{a,i}(s_1)\psi_{k'}^{b,i}(s_2)$. Let $E_j = \{\vec{x} \in \mathbb{Z}^4 : \rho_j(\vec{x}) \neq (0,0)\}$ and $E = E_1 \cap E_2$. Then,

$$\begin{aligned}
\chi_E(\vec{s},\vec{t}) &= \chi_{\mathbb{N}^2-(0,0)}(s_1,t_1)\chi_{\mathbb{N}^2-(0,0)}(s_2,t_2) \\
&= \left(\sum_{a=1}^{3}\sum_{k=1}^{\infty}\widehat{\psi_k^{a,1}}(s_1)\widehat{\psi_k^{a,2}}(t_1)\widehat{\psi_k^{a,3}}(-s_1-t_1)\right)\times \\
&\qquad \left(\sum_{b=1}^{3}\sum_{k'=1}^{\infty}\widehat{\psi_{k'}^{b,1}}(s_2)\widehat{\psi_{k'}^{b,2}}(t_2)\widehat{\psi_{k'}^{b,3}}(-s_2-t_2)\right) \\
&= \sum_{a,b=1}^{3}\sum_{k,k'=1}^{\infty}\widehat{\psi_{k,k'}^{a,b,1}}(\vec{s})\widehat{\psi_{k,k'}^{a,b,2}}(\vec{t})\widehat{\psi_{k,k'}^{a,b,3}}(-\vec{s}-\vec{t}).
\end{aligned}$$

Lemma 6.5. *Let $m : \mathbb{R}^4 \to \mathbb{C}$ be a bi-parameter multiplier and $\psi_{k,k'}^{a,b,1}, \psi_{k,k'}^{a,b,2}$ the functions in Remark 6.4. For every $k, k' \in \mathbb{N}$ and $1 \leq a, b \leq 3$, there is a smooth function $m_{a,b,k,k'}$ satisfying $m_{a,b,k,k'}(\vec{s},\vec{t})\widehat{\psi_{k,k'}^{a,b,1}}(\vec{s})\widehat{\psi_{k,k'}^{a,b,2}}(\vec{t}) = m(\vec{s},\vec{t})\widehat{\psi_{k,k'}^{a,b,1}}(\vec{s})\widehat{\psi_{k,k'}^{a,b,2}}(\vec{t})$ and*

$$m_{a,b,k,k'}(\vec{s},\vec{t}) = \sum_{\vec{n}\in\mathbb{Z}^4} c_{a,b,k,k',\vec{n}}e^{-2\pi i 2^{-k}\rho_1(\vec{n})\cdot(s_1,t_1)}e^{-2\pi i 2^{-k'}\rho_2(\vec{n})\cdot(s_2,t_2)},$$

where $|c_{a,b,k,k',\vec{n}}| \lesssim \prod_{j=1}^{4}(|n_j|+1)^{-5}$ uniformly in a, b, k, k'.

Proof. For simplicity, assume $a = b = 1$. Let $\varphi : \mathbb{R}^4 \to \mathbb{C}$ be a smooth function with

$$\operatorname{supp}(\varphi) \subseteq [-2^{-1}, 2^{-1}]^2 \times \Big([-2^{-1}, -2^{-11}] \cup [2^{-11}, 2^{-1}]\Big)^2 \quad \text{and}$$
$$\varphi = 1 \text{ on } [-2^{-2}, 2^{-2}]^2 \times \Big([-2^{-2}, -2^{-10}] \cup [2^{-10}, 2^{-2}]\Big)^2.$$

Define $m_{a,b,k,k'}(\vec{s}, \vec{t}) = m(\vec{s}, \vec{t})\varphi(2^{-k}s_1, 2^{-k'}s_2, 2^{-k}t_1, 2^{-k'}t_2)$. Then by construction, $m_{a,b,k,k'}(\vec{s}, \vec{t})\widehat{\psi_{k,k'}^{a,b,1}}(\vec{s})\widehat{\psi_{k,k'}^{a,b,2}}(\vec{t}) = m(\vec{s}, \vec{t})\widehat{\psi_{k,k'}^{a,b,1}}(\vec{s})\widehat{\psi_{k,k'}^{a,b,2}}(\vec{t})$. Further, if $E_{a,b,k,k'}$ is the support of $m_{a,b,k,k'}$, then $|E_{a,b,k,k'}| \leq 2^{2k}2^{2k'}$.

Recall that $\{2^{-k/2}e^{-2\pi i n 2^{-k}x}\}$ is an orthonormal basis on any interval of length 2^k, so that $\{2^{-k}e^{-2\pi i 2^{-k}\vec{n}\cdot\vec{x}}\}$ is an orthonormal basis on any square of side length 2^k. Thus,

$$m_{a,b,k,k'}(\vec{s}, \vec{t}) = \sum_{\vec{n}\in\mathbb{Z}^4} c_{a,b,k,k',\vec{n}}\, e^{-2\pi i 2^{-k}\rho_1(\vec{n})\cdot(s_1,t_1)}\, e^{-2\pi i 2^{-k'}\rho_2(\vec{n})\cdot(s_2,t_2)},$$

where $c_{a,b,k,k',\vec{n}}$ is

$$2^{-2k}2^{-2k'}\left(\int_{\mathbb{R}^4} m_{a,b,k,k'}(\vec{x}, \vec{y})\, e^{2\pi i 2^{-k}\rho_1(\vec{n})\cdot(x_1,y_1)}\, e^{2\pi i 2^{-k'}\rho_2(\vec{n})\cdot(x_2,y_2)}\, d\vec{x}\, d\vec{y}\right).$$

We may assume that if $\vec{n} = (n_1, n_2, n_3, n_4)$, each of n_j is nonzero, as these cases are handled similarly. Let $C = \max\{\|\partial^\alpha\varphi\|_\infty : 0 \leq |\alpha| \leq 20\}$. Note, if $(\vec{x}, \vec{y}) \in E_{a,b,k,k'}$, then $\|(x_1, y_1)\| \geq 2^{k-11}$ and $\|(x_2, y_2)\| \geq 2^{k'-11}$. So, $|\partial^{(\alpha,\beta)}m(\vec{x}, \vec{y})| \lesssim \|(x_1, y_1)\|^{-\alpha_1-\beta_1}\|(x_2, y_2)\|^{-\alpha_2-\beta_2} \lesssim 2^{-k(\alpha_1+\beta_1)}2^{-k'(\alpha_2+\beta_2)}$ for all $|\alpha|, |\beta| \leq 10$. Set $\gamma = (5, 5, 5, 5)$, and observe

$$\begin{aligned}
&|\partial^\gamma m_{a,b,k,k'}(\vec{x}, \vec{y})| \\
&\qquad \lesssim \sum_{(\alpha,\beta)\leq\gamma} |\partial^{(\alpha,\beta)}m_{a,b,k,k'}(\vec{x}, \vec{y})||\partial^{\gamma-(\alpha,\beta)}\varphi(2^{-k}x_1, 2^{-k'}x_2, 2^{-k}y_1, 2^{-k'}y_2)| \\
&\qquad \leq \sum_{(\alpha,\beta)\leq\gamma} C|\partial^{\alpha,\beta}m(\vec{x}, \vec{y})|2^{-k(10-\alpha_1-\beta_1)}2^{-k'(10-\alpha_2-\beta_2)} \\
&\qquad \lesssim 2^{-10k}2^{-10k'}.
\end{aligned}$$

By several iterations of integration by parts,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^4} m_{a,b,k,k'}(\vec{x},\vec{y})\, e^{2\pi i 2^{-k}\rho_1(\vec{n})\cdot(x_1,y_1)}\, e^{2\pi i 2^{-k'}\rho_2(\vec{n})\cdot(x_2,y_2)}\, d\vec{x}\, d\vec{y} \right| \\
&= \left| \int_{E_{a,b,k,k'}} \partial^\gamma m_{a,b,k,k'}(\vec{x},\vec{y})\, \frac{e^{2\pi i 2^{-k}\rho_1(\vec{n})\cdot(x_1,y_1)}\, e^{2\pi i 2^{-k'}\rho_2(\vec{n})\cdot(x_2,y_2)} (2\pi i)^{-20}}{(n_1 2^{-k})^5 (n_2 2^{-k'})^5 (n_3 2^{-k})^5 (n_4 2^{-k'})^5}\, d\vec{x}\, d\vec{y} \right| \\
&\lesssim \frac{2^{10k} 2^{10k'}}{|n_1|^5 |n_2|^5 |n_3|^5 |n_4|^5} |E_{a,b,k,k'}| \|\partial^\gamma m_{a,b,k,k'}\|_\infty \lesssim 2^{2k} 2^{2k'} \prod_{j=1}^{4} (|n_j|+1)^{-5}.
\end{aligned}$$

Namely, $|c_{a,b,k,k',\vec{n}}| \lesssim \prod(|n_j|+1)^{-5}$. To handle the cases when $n_j = 0$ for some j, adjust the above argument with $\gamma = (0,5,5,5)$ or $\gamma = (5,0,5,5)$, and so on. For $(a,b) \neq (1,1)$, we simply need to choose φ differently. □

Theorem 6.6. *For any bi-parameter multiplier m on $\mathbb{R}^4$, $\Lambda_m^{(2)} : L^{p_1} \times L^{p_2} \to L^p$ for $1 < p_1, p_2 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. If p_1 or p_2 or both are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$ and L^{p_j} replaced by $L \log L$.*

Proof. Fix m and let $f, g : \mathbb{T}^2 \to \mathbb{C}$. Then,

$$\Lambda_m^{(2)}(f,g)(\vec{x}) = \sum_{\vec{s},\vec{t} \in \mathbb{Z}^2} m(\vec{s},\vec{t}) \widehat{f}(\vec{s}) \widehat{g}(\vec{t}) e^{2\pi i \vec{x}\cdot(\vec{s}+\vec{t})}.$$

As in the proofs of Theorem 5.3 and Theorem 5.8, we can assume $m(\vec{0},\vec{0}) = 0$.

Let $m_1(s_1,t_1) = m(s_1,0,t_1,0)$. Then, $m_1 : \mathbb{R}^2 \to \mathbb{C}$ is a Coifman-Meyer multiplier. Let $F_1(x_1) = \int_{\mathbb{T}} f(x_1,x_2)\, dx_2$ and $G_1(x_1) = \int_{\mathbb{T}} g(x_1,x_2)\, dx_2$, so that $\widehat{f}(s_1,0) = \widehat{F_1}(s_1)$ and $\widehat{g}(t_1,0) = \widehat{G_1}(t_1)$. Let

$$\Lambda_{m_1}(F_1,G_1)(x) = \sum_{s_1,t_1 \in \mathbb{Z}} m_1(s_1,t_1) \widehat{F_1}(s_1) \widehat{G_1}(t_1) e^{2\pi i x(s_1+t_1)},$$

a standard Coifman-Meyer operator. Now define $m_2(s_2,t_2) = m(0,s_2,0,t_2)$ and F_2, G_2 as expected. Let $\Lambda_{m_2}(F_2,G_2)(y)$ be the appropriate Coifman-Meyer operator. Finally, let m_0 be a bi-parameter multiplier which agrees with m on integers away from the planes $\{(s_1,t_1)=0\}$ and $\{(s_2,t_2)=0\}$, but is 0 on these planes. Then,

$$\Lambda_m^{(2)}(f,g)(x,y)=\Lambda_{m_0}^{(2)}(f,g)(x,y)+\Lambda_{m_1}(F_1,G_1)(x)+\Lambda_{m_2}(F_2,G_2)(y).$$

By Theorem 5.8, if $p_1,p_2>1$, then $\|\Lambda_{m_1}(F_1,G_1)\|_{L^p(\mathbb{T})}\lesssim\|F_1\|_{L^{p_1}(\mathbb{T})}\|G_1\|_{L^{p_2}(\mathbb{T})}$. By generalized Minkowski, $\|F_1\|_{L^{p_1}(\mathbb{T})}\leq\|f\|_{p_1}$ and $\|G_1\|_{L^{p_2}(\mathbb{T})}\leq\|g\|_{p_2}$. So, $\|\Lambda_{m_1}(F_1,G_1)\|_{L^p(\mathbb{T})}\lesssim\|f\|_{p_1}\|g\|_{p_2}$. If $p_1=1$, then $\|F_1\|_1\leq\|f\|_1\leq\|f\|_{L\log L}$. Similarly for $p_2=1$. Thus, the term $\Lambda_{m_1}(F_1,G_1)$, and by symmetry $\Lambda_{m_2}(F_2,G_2)$, satisfies all the estimates we want. Hence, it suffices to consider the operator Λ_{m_0}. Equivalently, we can assume m is 0 on the planes $\{(s_1,t_1)=0\}$ and $\{(s_2,t_2)=0\}$.

Let $h\in L^1(\mathbb{T}^2)$. Let $f_0=\widetilde{\widetilde{f}}$ and similarly for g_0,h_0. Then,

$$\begin{aligned}
\langle\Lambda_m^{(2)}(f,g),\widetilde{h}\rangle&=\int_{\mathbb{T}^2}\Lambda_m^{(2)}(f,g)(x)h_0(x)\,dx\\
&=\int_{\mathbb{T}^2}\left(\sum_{\vec{s},\vec{t}\in\mathbb{Z}^2}^{\infty}m(\vec{s},\vec{t})\widehat{f}(\vec{s})\widehat{g}(\vec{t})e^{2\pi i\vec{x}(\vec{s}+\vec{t})}\right)h_0(\vec{x})\,d\vec{x}\\
&=\sum_{\vec{s},\vec{t}\in\mathbb{Z}^2}m(\vec{s},\vec{t})\widehat{f}(\vec{s})\widehat{g}(\vec{t})\int_{\mathbb{T}^2}h_0(\vec{x})e^{2\pi i\vec{x}(\vec{s}+\vec{t})}\,d\vec{x}\\
&=\sum_{\vec{s},\vec{t}\in\mathbb{Z}^2}m(\vec{s},\vec{t})\widehat{f}(\vec{s})\widehat{g}(\vec{t})\widehat{h_0}(-\vec{s}-\vec{t}).
\end{aligned}$$

Now employ Remark 6.4 to write

$$\begin{aligned}
&\langle\Lambda_m^{(2)}(f,g),\widetilde{h}\rangle\\
&=\sum_{a,b=1}^{3}\sum_{k,k'=1}^{\infty}\sum_{\vec{s},\vec{t}\in\mathbb{Z}^2}m(\vec{s},\vec{t})\widehat{f}(\vec{s})\widehat{\psi_{k,k'}^{a,b,1}}(\vec{s})\widehat{g}(\vec{t})\widehat{\psi_{k,k'}^{a,b,2}}(\vec{t})\widehat{h_0}(-\vec{s}-\vec{t})\widehat{\psi_{k,k'}^{a,b,3}}(-\vec{s}-\vec{t})\\
&=\sum_{a,b=1}^{3}\sum_{k,k'=1}^{\infty}\sum_{\vec{s},\vec{t}\in\mathbb{Z}^2}m_{a,b,k,k'}(\vec{s},\vec{t})\widehat{f}(\vec{s})\widehat{\psi_{k,k'}^{a,b,1}}(\vec{s})\widehat{g}(\vec{t})\widehat{\psi_{k,k'}^{a,b,2}}(\vec{t})\widehat{h_0}(-\vec{s}-\vec{t})\widehat{\psi_{k,k'}^{a,b,3}}(-\vec{s}-\vec{t})\\
&=:\sum_{a,b=1}^{3}S_{a,b},
\end{aligned}$$

where $m_{a,b,k,k'}$ is as given in Lemma 6.5. Let $\psi^{a,b,1}_{k,k',\vec{n}_1}(\vec{x}) = \psi^{a,b,1}_{k,k'}(\vec{x}-(2^{-k}n_1, 2^{-k'}n_2))$ and $\psi^{a,b,2}_{k,k',\vec{n}_2}(\vec{x}) = \psi^{a,b,2}_{k,k'}(\vec{x}-(2^{-k}n_3, 2^{-k'}n_4)))$. Then,

$$
\begin{aligned}
S_{a,b} &= \sum_{\vec{n}\in\mathbb{Z}^4}\sum_{k,k'=1}^{\infty}\sum_{\vec{s},\vec{t}\in\mathbb{Z}^2} c_{a,b,k,k',\vec{n}}(f*\psi^{a,b,1}_{k,k',\vec{n}_1})\widehat{\ }(\vec{s})(g*\psi^{a,b,2}_{k,k',\vec{n}_2})\widehat{\ }(\vec{t})(h_0*\psi^{a,b,3}_{k,k'})\widehat{\ }(-\vec{s}-\vec{t}) \\
&= \sum_{\vec{n}\in\mathbb{Z}^4}\sum_{k,k'=1}^{\infty} c_{a,b,k,k',\vec{n}}\int_{\mathbb{T}^2}(f*\psi^{a,b,1}_{k,k',\vec{n}_1})(\vec{x})(g*\psi^{a,b,2}_{k,k,\vec{n}_2})(\vec{x})(h_0*\psi^{a,b,3}_{k,k'})(\vec{x})\,d\vec{x}.
\end{aligned}
$$

The last line is gained from showing Claim 5.6 is valid on $\mathbb{T}^d$ for any d. Just as in the previous proofs, we can dilate and translate to write

$$
\begin{aligned}
&\int_{\mathbb{T}^2}(f*\psi^{a,b,1}_{k,k',\vec{n}_1})(\vec{x})(g*\psi^{a,b,2}_{k,k',\vec{n}_2})(\vec{x})(h_0*\psi^{a,b,3}_{k,k'})(\vec{x})\,d\vec{x} \\
&\qquad = 2^{-k}2^{-k'}\sum_{j=0}^{2^k-1}\sum_{j'=0}^{2^{k'}-1}\int_{[0,1]^2}\langle\psi^{a,b,1}_{k,k',j,j',\vec{n}_1,\vec{\alpha}},\overline{f}\rangle\langle\psi^{a,b,2}_{k,k',j,j',\vec{n}_2,\vec{\alpha}},\overline{g}\rangle\langle\psi^{a,b,3}_{k,k',j,j',\vec{\alpha}},\overline{h_0}\rangle\,d\vec{\alpha},
\end{aligned}
$$

where $\psi^{a,b,1}_{k,k',j,j',\vec{n}_1,\vec{\alpha}}(\vec{x}) = \psi^{a,b,1}_{k,k'}(2^{-k}(\alpha_1+j+n_1)-x_1, 2^{-k'}(\alpha_2+j'+n_2)-x_2)$, and similarly for the other two functions.

For a dyadic rectangle $R = [2^{-k}j, 2^{-k}(j+1)]\times[2^{-k'}j', 2^{-k'}(j'+1)]$, let $\varphi^{a,b,1}_{R^{\vec{n}_1}_{\vec{\alpha}}}$ be the reflection of $2^{-k}2^{-k'}\psi^{a,b,1}_{k,k',j,j',\vec{n}_1,\vec{\alpha}}$, and similarly for $\varphi^{a,b,2}_{R^{\vec{n}_2}_{\vec{\alpha}}}$ and $\varphi^{a,b,3}_{R_{\vec{\alpha}}}$. It is easily checked that the construction of $\psi^{a,b,i}_{k,k'}$ guarantees that $\varphi^{a,b,i}_R$ are the tensor products of adapted families with $\int_{\mathbb{T}}\varphi^{a,b,i}_I\,dx = 0$ when $a\neq i$ and $\int_{\mathbb{T}}\varphi^{a,b,i}_J\,dx = 0$ when $b\neq i$. Let $\phi^{a,b,i}_R = |R|^{-1/2}\varphi^{a,b,i}_R$, so that

$$
S_{a,b} = \sum_{\vec{n}\in\mathbb{Z}^4}\int_{[0,1]^2}\sum_R c_{a,b,R,\vec{n}}\frac{1}{|R|^{1/2}}\langle\phi^{a,b,1}_{R^{\vec{n}_1}_{\vec{\alpha}}},f_0\rangle\langle\phi^{a,b,2}_{R^{\vec{n}_2}_{\vec{\alpha}}},g_0\rangle\langle\phi^{a,b,3}_{R_{\vec{\alpha}}},h\rangle\,d\alpha,
$$

where the inner sum is over all dyadic rectangles and $c_{a,b,R,\vec{n}} = c_{a,b,k,k',\vec{n}}$ when $R = I\times J$ with $|I| = 2^{-k}, |J| = 2^{-k'}$. Write $c'_{a,b,R,\vec{n}} = \prod_{j=1}^4(|n_j|+1)^5 c_{a,b,R,\vec{n}}$, which are uniformly bounded in R and $\vec{n}$ by Lemma 6.5. Hence,

$$S_{a,b} = \sum_{\vec{n}\in\mathbb{Z}^4} \prod_{j=1}^{4} \frac{1}{(|n_j|+1)^5} \int_{[0,1]} \sum_{R} c'_{a,b,R,\vec{n}} \frac{1}{|R|^{1/2}} \langle \phi^{a,b,1}_{R^{\vec{n_1}}_{\vec{\alpha}}} f_0\rangle \langle \phi^{a,b,2}_{R^{\vec{n_2}}_{\vec{\alpha}}}, g_0\rangle \langle \phi^{a,b,2}_{R_{\vec{\alpha}}}, h\rangle \, d\alpha$$
$$= \Big\langle \sum_{\vec{n}\in\mathbb{Z}^4} \prod_{j=1}^{4} \frac{1}{(|n_j|+1)^5} T^{a,b,[\vec{n}]}_{c'}(f_0, g_0), h \Big\rangle$$

As $h \in L^1$ is arbitrary, it follows that

$$\widetilde{\Lambda^{(2)}_m(f,g)} = \sum_{\vec{n}\in\mathbb{Z}^4} \prod_{j=1}^{4} \frac{1}{(|n_j|+1)^5} \sum_{a,b=1}^{3} T^{a,b,[\vec{n}]}_{c'}(f_0, g_0)$$

almost everywhere. We know $\|T^{a,b,[\vec{n}]}_{c'}(f_0,g_0)\|_p \lesssim \prod_j(|n_j|+1)\|f\|_{p_1}\|g\|_{p_2}$ when $p_1, p_2 > 1$. So, $\|\Lambda^{(2)}_m(f,g)\|_p \lesssim \|f\|_{p_1}\|g\|_{p_2}$ whenever $p \geq 1$ follows immediately. By Lemma 5.1 (with $k = 2$), $\|\Lambda^{(2)}_m(f,g)\|_{p,\infty} \lesssim \|f\|_{p_1}\|g\|_{p_2}$ for all $p_1, p_2 > 1$. By interpolation of these cases, $\|\Lambda^{(2)}_m(f,g)\|_p \lesssim \|f\|_{p_1}\|g\|_{p_2}$ whenever $p_1, p_2 > 1$ and $p < 1$.

On the other hand, $\|T^{a,b,[\vec{n}]}_{c'}(f_0,g_0)\|_{p,\infty} \lesssim \prod_j(|n_j|+1)\|f\|_{L\log L}\|g\|_{p_2}$ whenever $p_1 = 1$. By applying Lemma 5.1 again, $\|\Lambda^{(2)}_m(f,g)\|_{p,\infty} \lesssim \|f\|_{L\log L}\|g\|_{p_2}$. The cases $p_2 = 1$ and $p_1 = p_2 = 1$ follow in the same way. $\square$

Chapter 7
Multi-parameter Multipliers

Finally, we would like to consider multipliers, and their corresponding operators, which are multi-parameter. That is, m acts as if the product of s Coifman-Meyer multipliers.

For a vector $\vec{t} \in \mathbb{R}^{sd}$, let $\rho_j(\vec{t}) = (t_j, t_{j+s}, \ldots, t_{j+s(d-1)}) \in \mathbb{R}^d$ for $1 \le j \le s$. Conversely, for $1 \le j \le d$, let $\vec{t_j} = (t_{s(j-1)+1}, \ldots, t_{js}) \in \mathbb{R}^s$ so that $\vec{t} = (\vec{t_1}, \ldots, \vec{t_d})$.

Let $m : \mathbb{R}^{sd} \to \mathbb{C}$ be smooth away from the origin and uniformly bounded. We say m is an s-parameter multiplier if

$$|\partial^\alpha m(\vec{t})| \lesssim \prod_{j=1}^{s} \|\rho_j(\vec{t})\|^{-|\rho_j(\alpha)|}$$

for all indices $|\alpha| \le sd(d+3)$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$.

Given such a multiplier m on $\mathbb{R}^{sd}$ and L^1 functions $f_1, \ldots, f_d : \mathbb{T}^s \to \mathbb{C}$, we define the associated multiplier operator $\Lambda_m^{(s)}(f_1, \ldots, f_d) : \mathbb{T}^s \to \mathbb{C}$ as

$$\Lambda_m^{(s)}(f_1, \ldots, f_d)(\vec{x}) = \sum_{\vec{t} \in \mathbb{Z}^{sd}} m(\vec{t}) \widehat{f_1}(\vec{t_1}) \cdots \widehat{f_d}(\vec{t_d}) e^{2\pi i \vec{x} \cdot (\vec{t_1} + \ldots + \vec{t_d})}.$$

The L^p estimates of previous chapters still hold with minor modifications.

Theorem 7.1. *For any s-parameter multiplier m on $\mathbb{R}^{sd}$, $\Lambda_m^{(s)} : L^{p_1} \times \ldots \times L^{p_d} \to L^p$ for $1 < p_j < \infty$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_d} = \frac{1}{p}$. If any or all of the p_j are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$ and L^{p_j} replaced by $L(\log L)^{s-1}$. In particular, $\Lambda_m^{(s)} : L(\log L)^{s-1} \times \ldots \times L(\log L)^{s-1} \to L^{1/d,\infty}$.*

In view of all the results, we now have a good view of the heuristics. Away from $p_j = 1$, each of these operators act the same. However, it is these endpoint cases which are the most interesting. Each time we go up a parameter, we "gain a log" at the endpoint.

It will not be our goal in this chapter to explicitly prove this result. Indeed, it should be clear that the method of proof employed on increasing complex multiplier operators throughout this text can be used. Instead, we give a brief survey of how the argument would go.

By induction, we can assume this theorem is known for $(s-1)$-parameter multipliers. Like in the proof of Theorem 6.6, this allows us to assume m is 0 on the planes $\{\rho_j(\vec{t}) = 0\}$. Then, we can introduce bump functions which are the s-fold tensor products of the functions in Theorem 1.6 (as the functions in Remark 6.4 are the 2-fold tensor products). By the same dilation and translations, our problem boils to understanding the appropriate paraproducts.

We say $Q \subset \mathbb{T}^s$ is a dyadic rectangle if $Q = I_1 \times \ldots \times I_s$ for dyadic intervals I_j. Define $\varphi_Q : \mathbb{T}^s \to \mathbb{C}$ to be the s-fold tensor product of adapted families. The appropriate (bilinear) paraproducts in this setting are

$$T_\epsilon^{a_1,\ldots,a_s}(f,g)(\vec{x}) = \sum_Q \epsilon_Q \frac{1}{|Q|^{1/2}} \langle \phi_Q^1, f\rangle \langle \phi_Q^2, g\rangle \phi_Q^3(\vec{x})$$

where the sum is over all dyadic rectangles Q and (ϵ_Q) is a uniformly bounded sequence. Each a_j ranges over $1, 2, 3$. If $\phi_Q^i = \phi_{I_1}^i \oplus \ldots \oplus \phi_{I_s}^i$, then $\int_{\mathbb{T}} \phi_{I_j}^i \, dx = 0$ whenever $i \neq a_j$.

To complete the proof on s-parameter multiplier operators, it suffices to show the associated paraproducts satisfy the same bounds. The stopping time argument presented in Theorems 3.10, 5.5, and 6.3 works equally well in all dimensions, given the correct s-fold hybrid operators. For example, when $s = 3$, we consider SSS, SSM, MSM, etc. Therefore, we will understand the paraproducts if we can show each s-fold hybrid operator maps $L^p \to L^p$ for $1 < p < \infty$ and $L(\log L)^{s-1} \to L^{1,\infty}$.

For illustrative purposes, we show this for three specific operators when $s = 3$. For $f : \mathbb{T}^3 \to \mathbb{C}$ define

$$SSSf(x,y,z) = \Big(\sum_Q \frac{|\langle \phi_Q, f\rangle|^2}{|Q|} \chi_Q(x,y,z) \Big)^{1/2},$$

$$SSMf(x,y,z) = \Big(\sum_{I_1} \sum_{I_2} \frac{\Big(\sup_{I_3} \frac{1}{|I_3|^{1/2}} |\langle \phi_Q, f\rangle| \chi_{I_3}(z) \Big)^2}{|I_1||I_2|} \chi_{I_1}(x)\chi_{I_2}(y) \Big)^{1/2},$$

and

$$SMMf(x,y,z) = \Big(\sum_{I_1} \frac{\Big(\sup_{I_2} \sup_{I_3} \frac{1}{|I_2|^{1/2}} \frac{1}{|I_3|^{1/2}} |\langle \phi_Q, f\rangle| \chi_{I_2}(y)\chi_{I_3}(z) \Big)^2}{|I_1|} \chi_{I_1}(x) \Big)^{1/2}.$$

Start with $SSSf$. Using the same notational conveniences as before,

$$\begin{aligned} SSSf &= \Big(\sum_{I_1} \sum_{I_2} \sum_{I_3} \frac{1}{|I_3|} \Big| \Big\langle \phi_{I_3}, \frac{\langle f, \phi_{I_1} \oplus \phi_{I_2} \rangle}{|I_1|^{1/2}|I_2|^{1/2}} \chi_{I_1}\chi_{I_2} \Big\rangle \Big|^2 \chi_{I_3} \Big)^{1/2} \\ &= \Big(\sum_{I_1} \sum_{I_2} S_3 \Big(\frac{\langle f, \phi_{I_1} \oplus \phi_{I_2} \rangle}{|I_1|^{1/2}|I_2|^{1/2}} \chi_{I_1}\chi_{I_2} \Big)^2 \Big)^{1/2}. \end{aligned}$$

So,

$$\begin{aligned} \|SSSf\|_p &= \Big\| \Big(\sum_{I_1} \sum_{I_2} S_3 \Big(\frac{\langle f, \phi_{I_1} \oplus \phi_{I_2} \rangle}{|I_1|^{1/2}|I_2|^{1/2}} \chi_{I_1}\chi_{I_2} \Big)^2 \Big)^{1/2} \Big\|_p \\ &\lesssim \Big\| \Big(\sum_{I_1} \sum_{I_2} \frac{|\langle f, \phi_{I_1} \oplus \phi_{I_2} \rangle|^2}{|I_1||I_2|} \chi_{I_1}\chi_{I_2} \Big)^{1/2} \Big\|_p \\ &= \Big\| \Big(\sum_{I_1} S_2 \Big(\frac{\langle f, \phi_{I_1} \rangle}{|I_1|^{1/2}} \chi_{I_1} \Big)^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_{I_1} \frac{|\langle f, \phi_{I_1} \rangle|^2}{|I_1|} \chi_{I_1} \Big)^{1/2} \Big\|_p \\ &= \|S_1 f\|_p \lesssim \|f\|_p, \end{aligned}$$

and

$$
\begin{aligned}
\|SSSf\|_{1,\infty} &= \Big\|\Big(\sum_{I_1}\sum_{I_2} S_3\Big(\frac{\langle f, \phi_{I_1}\oplus\phi_{I_2}\rangle}{|I_1|^{1/2}|I_2|^{1/2}}\chi_{I_1}\chi_{I_2}\Big)^2\Big)^{1/2}\Big\|_{1,\infty} \\
&\lesssim \Big\|\Big(\sum_{I_1} S_2\Big(\frac{\langle f, \phi_{I_1}\rangle}{|I_1|^{1/2}}\chi_{I_1}\Big)^2\Big)^{1/2}\Big\|_{1} \lesssim \|S_1 f\|_{L\log L} \lesssim \|f\|_{L(\log L)^2}.
\end{aligned}
$$

Using the same kind of argument

$$
\begin{aligned}
\|SSMf\|_p &= \Big\|\Big(\sum_{I_1}\sum_{I_2} M_3'\Big(\frac{\langle f, \phi_{I_1}\oplus\phi_{I_2}\rangle}{|I_1|^{1/2}|I_2|^{1/2}}\chi_{I_1}\chi_{I_2}\Big)^2\Big)^{1/2}\Big\|_{p} \\
&\lesssim \Big\|\Big(\sum_{I_1}\sum_{I_2} \frac{|\langle f, \phi_{I_1}\oplus\phi_{I_2}\rangle|^2}{|I_1||I_2|}\chi_{I_1}\chi_{I_2}\Big)^{1/2}\Big\|_{p} \\
&= \Big\|\Big(\sum_{I_1} S_2\Big(\frac{\langle f, \phi_{I_1}\rangle}{|I_1|^{1/2}}\chi_{I_1}\Big)^2\Big)^{1/2}\Big\|_{p} \lesssim \Big\|\Big(\sum_{I_1} \frac{|\langle f, \phi_{I_1}\rangle|^2}{|I_1|}\chi_{I_1}\Big)^{1/2}\Big\|_{p} \\
&= \|S_1 f\|_p \lesssim \|f\|_p,
\end{aligned}
$$

and

$$
\begin{aligned}
\|SSSf\|_{1,\infty} &= \Big\|\Big(\sum_{I_1}\sum_{I_2} M_3'\Big(\frac{\langle f, \phi_{I_1}\oplus\phi_{I_2}\rangle}{|I_1|^{1/2}|I_2|^{1/2}}\chi_{I_1}\chi_{I_2}\Big)^2\Big)^{1/2}\Big\|_{1,\infty} \\
&\lesssim \Big\|\Big(\sum_{I_1} S_2\Big(\frac{\langle f, \phi_{I_1}\rangle}{|I_1|^{1/2}}\chi_{I_1}\Big)^2\Big)^{1/2}\Big\|_{1} \lesssim \|S_1 f\|_{L\log L} \lesssim \|f\|_{L(\log L)^2}.
\end{aligned}
$$

By the same method used in Theorem 6.1 for MM,

$$
\sup_{I_2}\sup_{I_3}\frac{1}{|I_2|^{1/2}}\frac{1}{|I_3|^{1/2}}|\langle\phi_Q, f\rangle|\chi_{I_1}\chi_{I_2}\chi_{I_3} \lesssim M_2\circ M_3(\langle\phi_{I_1}, f\rangle\chi_{I_1}).
$$

Thus,

$$
\begin{aligned}
\|SMMf\|_p &\lesssim \Big\|\Big(\sum_{I_1} M_2\circ M_3\Big(\frac{\langle f, \phi_{I_1}\rangle}{|I_1|^{1/2}}\chi_{I_1}\Big)^2\Big)^{1/2}\Big\|_{p} \\
&\lesssim \Big\|\Big(\sum_{I_1} M_3\Big(\frac{\langle f, \phi_{I_1}\rangle}{|I_1|^{1/2}}\chi_{I_1}\Big)^2\Big)^{1/2}\Big\|_{p} \\
&\lesssim \Big\|\Big(\sum_{I_1} \frac{|\langle f, \phi_{I_1}\rangle|^2}{|I_1|}\chi_{I_1}\Big)^{1/2}\Big\|_{p} = \|S_1 f\|_p \lesssim \|f\|_p,
\end{aligned}
$$

and

$$\begin{aligned}
\|SMMf\|_{1,\infty} &\lesssim \Big\| \Big(\sum_{I_1} M_2 \circ M_3 \Big(\frac{\langle f, \phi_{I_1} \rangle}{|I_1|^{1/2}} \chi_{I_1} \Big)^2 \Big)^{1/2} \Big\|_{1,\infty} \\
&\lesssim \Big\| \Big(\sum_{I_1} M_3 \Big(\frac{\langle f, \phi_{I_1} \rangle}{|I_1|^{1/2}} \chi_{I_1} \Big)^2 \Big)^{1/2} \Big\|_{1} \\
&\lesssim \Big\| \Big(\sum_{I_1} \frac{|\langle f, \phi_{I_1} \rangle|^2}{|I_1|} \chi_{I_1} \Big)^{1/2} \Big\|_{L \log L} = \|S_1 f\|_{L \log L} \lesssim \|f\|_{L(\log L)^2}.
\end{aligned}$$

We also have $MMMf \lesssim M_S f \leq M_1 \circ M_2 \circ M_3 f$, for which the desired estimates clearly hold. Finally, we note as before that SMS and MSS are pointwise smaller than a kind of SSM, while MMS and MSM are smaller than a kind of SMM.

The recipe for arbitrary s-fold hybrid operators should now be clear. It suffices to consider only the ones of the form $SS...SMM...M$. In this case, the $M...MM$ part is pointwise smaller than $M_j \circ M_{j+1} \circ \cdots \circ M_s$. Repeated iterations of Fefferman-Stein eliminate these M_j, while the remaining $SS...S$ part can be dealt with as usual.

In conclusion, Theorem 7.1 can be proven by the same methods presented in earlier chapters, with only minor adjustments here and there.

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